Rotation and Link Invariants

(ローテーションと絡み目不変量)

平 成 1 5 年 度

Makiko Ishiwata

(石 渡　万 希 子)

Contents

1 Introduction

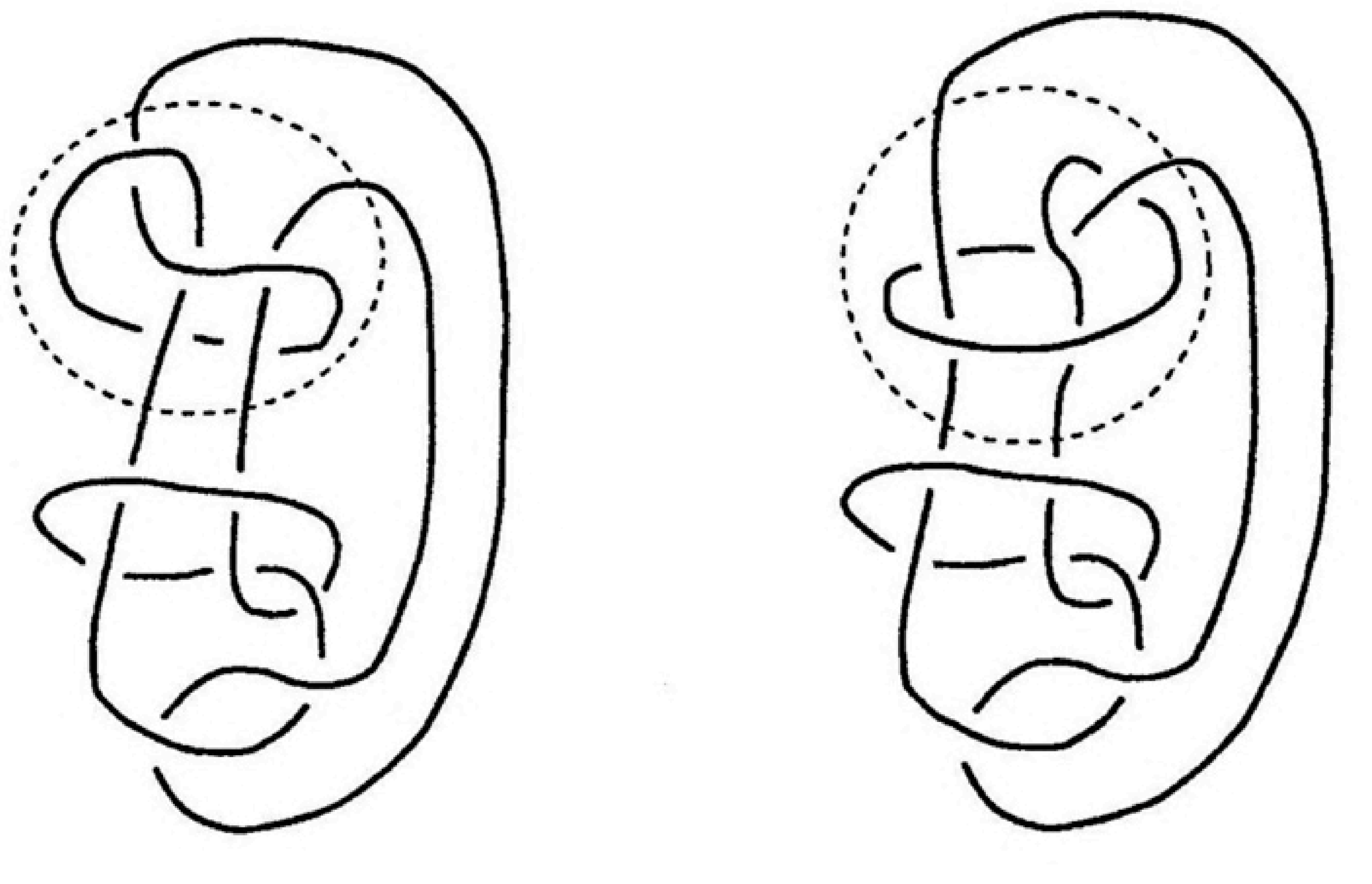
An n-component link is the image of a smooth embedding of disjoint union of n circles into S^3. In particular, a 1-component link is a knot. A main purpose of Knot Theory is a classification of link types (i.e., isotopy classes of links) as a mathematical object.

Link invariants play an important role to distinguish two knots. One of the most classical invariant of links is the Alexander polynomial. It was due to J.Alexander in about 1928 and which is derived from homology of the infinite cyclic covering space of a link complement. The computation of algorithmic procedures are easy to master and it was one of the most useful invariant. Knot theorists utilized mainly the Alexander polynomial to distinguish links for over 50 years.

However, the Alexander polynomial remains a lot of links not to be distinguished. In 1984, V.F.R. Jones invented a new powerful polynomial invariant which is called the Jones polynomial. The Alexander polynomial does not distinguish, for example, right handed trefoil from left handed trefoil, square knot from granny knot, and the Jones polynomial distinguishes them.

Since the big discovery of Jones polynomial invariant, many researchers have attempted to discover new polynomial invariants. Immediately after discovery of Jones, another polynomial invariant, so-called Homflypt polynomial was invented in [10, 33]. This polynomial is the generalization of either Jones polynomial or the Alexander polynomial with two variables. In fact, Homflypt polynomial is better polynomial invariant than both Jones polynomial and Alexander polynomial, since the both of those polynomials are simply special case of Homflypt polynomial.

However, an efficient complete link invariant has not been discovered so far. In particular, the Homflypt polynomial is not complete link invariant, that is, there are different knots which share the same Homflypt polynomial. For example, Conway knot and Kinoshita-Terasaka knot (Fig.1.1) are famous. We remark this pair was distinguished by another link invariant, which is not polynomial invariant, the genera of knots. Although the genus of knot is rough invariant, it is helpful for this case. What an interesting phenomenon it is!



Conway knot Kinoshita-Terasaka knot

Fig. 1.1

Investigation of the precision of known knot invariants is required for classification of links. As a way of investigation, several researchers have constructed links which share the same knot invariants. T.Kanenobu constructed infinitely many different knots which share the same Homflypt (hence Jones) polynomials [19].

A simple method of producing different links with the same polynomial invariant is *mutation*. For example, the pair of Conway knot and Kinoshita-Terasaka knot (Fig.1.1), which are transformed each other by changing a tangle part with 2 strings, introduced by J.H.Conway before 1960. In fact, mutation of tangle is known to be a big trouble for skein polynomial invariants such as Homflypt polynomial. Mutation is also known as a deformation which preserve the other type of knot invariant, such as the homeomorphic type of the double branched cover of S^3 branched along the link.

As a generalization of mutation, the concept of *rotation* were adapted to knot theory by R.P.Anstee, J.H.Przytycki and D.Rolfsen [3] in 1987. Rotation of order n was defined as the local deformation on the tangle part of n-strings with n-rotational symmetry. The n-tangle part of rotation is called rotor and the concept was first introduced in graph theory by W.Tutte [5].

In [3] it was shown that rotation of order less than six (resp. five and four) preserves Jones polynomial (resp. Homflypt polynomial and Kauffman polynomial). In [17], G.T.Jin and Rolfsen showed these are the best possible. Especially, they showed a pair of 5-rotant links, illustrated in Fig.1.2 share the same Jones polynomial but they are distinguished by Homflypt polynomials (Fig.1.2 [17]). Now we can see the advantage of Homflypt polynomial over Jones polynomial concretely by mutation and rotation.

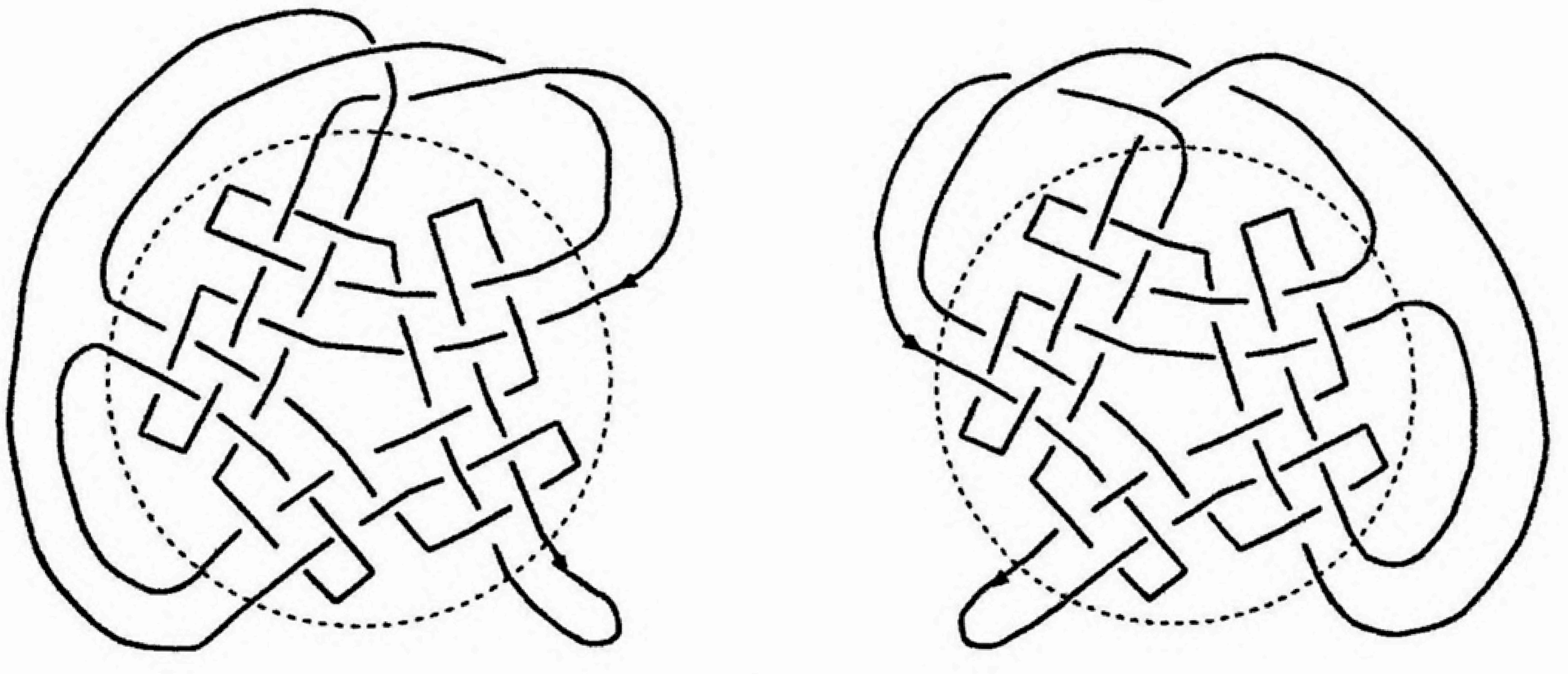


Fig. 1.2

Here, we pay attention to the result that rotation of order three or four preserves Alexander polynomial of links. In [17], Jin and Rolfsen gave the following question : Does the rotation preserve the Alexander polynomial? In 2001, P.Traczyk [38] showed that Alexander polynomials of a pair of any *orientation-preserving* rotants coincide. But it was inconclusive if any *orientation-reversing* rotations preserve Alexander polynomials. So the question is still open. In section 3, we give a negative answer for the open question. In fact, we present an example which shows that there is a pair of orientation-reversing rotants which do not share the same Alexander polynomial. The method of Traczyk's proof in [38] was concerned with rotation matrix to rotational symmetry of rotor part. It was quite impressive and the author was motivated to examine how rotations affect the knot invariants which are brought by quadratic forms associated to spanning surface of links. We also show, in section 3, that any knot invariant obtained from the characteristic polynomial of the Goeritz matrix, including Murasugi's signature, is not changed by any rotation. We show in particular that the Murasugi's unoriented version of the classical signature [13, 29, 30] (resp. Tristram-Levine signature [39]) is preserved by any rotations (resp. any orientation-preserving rotations) (see Theorems 3.2 and 3.4).

In section 4, we investigate tangles in another way. In particular, we construct a branched covering space associated with tangles. Tangles were first studied by Conway [7]. They were particularly useful for analyzing links as mentioned above. A branched cover of the three-ball branched along a tangle (succinctly a branched cover of a tangle) is an indispensable tool for understanding tangles. Hence it is important to give practical presentations of branched covers of tangles. Recall that a p-fold cyclic branched cover of a link or tangle (oriented for $p > 2$) is uniquely defined by an epimorphism of the fundamental group of the complement onto $\mathbb{Z}_p$ which sends meridians to 1. A p-fold branched cover of an n-tangle is a three-manifold, the boundary of which is a connected orientable surface of genus $(n-1)(p-1)$. Such a manifold can be obtained from the genus $(n-1)(p-1)$ handlebody by a surgery. We provide an algorithm for a surgery description of a p-fold cyclic branched cover of B^3 branched along a tangle. The construction generalizes that of J.M.Montesinos [27] and S.Akbulut and R.Kirby [2]. It is strikingly simple in the case of a two-fold branched cover.

As we construct the branched covering space of S^3 branched along a link by cutting-along a Seifert surface of the link, we proceed the construction of covering space associated with a tangle by cutting-along a certain surface in B^3. We introduce a disk-band representation of a tangle, which is connected orientable surface bounded by a tangle in B^3. We also show that any tangle has a disk-band representation. Obviously this disk-band representation plays a key role of our construction of branched covering of a tangle.

As an application of this construction, we examine the branched covering space of rotant links. In general, it is not true that a rotation preserves the first homology group of the double branched cover S^3 branched along a link. Necessary conditions for preservation of the homology group are given in [9]. However, if we assume that a pair of oriented rotants with "special" disk-band representation, then we conclude that the first homology groups of the double branched covers of S^3 branched along a pair of rotants are isotopic (see Theorem 4.4).

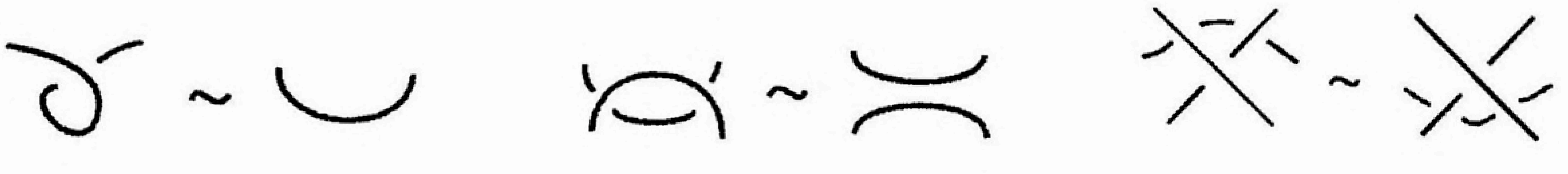
We also discuss the related Heegaard decomposition of a p-fold branched cover of an n-tangle. In addition to the surgery presentation, it is also useful to have another presentation of a p-fold cyclic branched cover. Our construction leads straightforwardly to a Heegaard decomposition, that is a decomposition into a *compression body* [4, 6] and a handlebody, of a p-fold cyclic branched cover.

Now, we mention another link invariant, which is called the genus of a link. Seifert surfaces for a given link are not uniquely determined, but we may regard the minimal genus of such Seifert surfaces as a link invariant. In fact, though Conway knot and Kinoshita-Terasaka knot share the same skein polynomial invariants, we saw the difference of these link types by their genera 3 and 2 respectively in [11]. D. Gabai determined the genera of prime links with up to 9 crossings in [11] and [12]. In appendix, we exhibit the minimal genus Seifert surfaces of prime n-component links ($n = 2, 3, 4$) with up to 9 crossings (at least one minimal genus Seifert surface for each link) (for the notation we refer [34, Appendix C]). We determined the genera of links by applying product decompositions and disk decompositions come from the sutured manifold theory ([11, Theorem 4.1]).

2 Preliminary

In this section, we describe briefly some basic terms of link invariants, and basic properties which are related to our arguments.

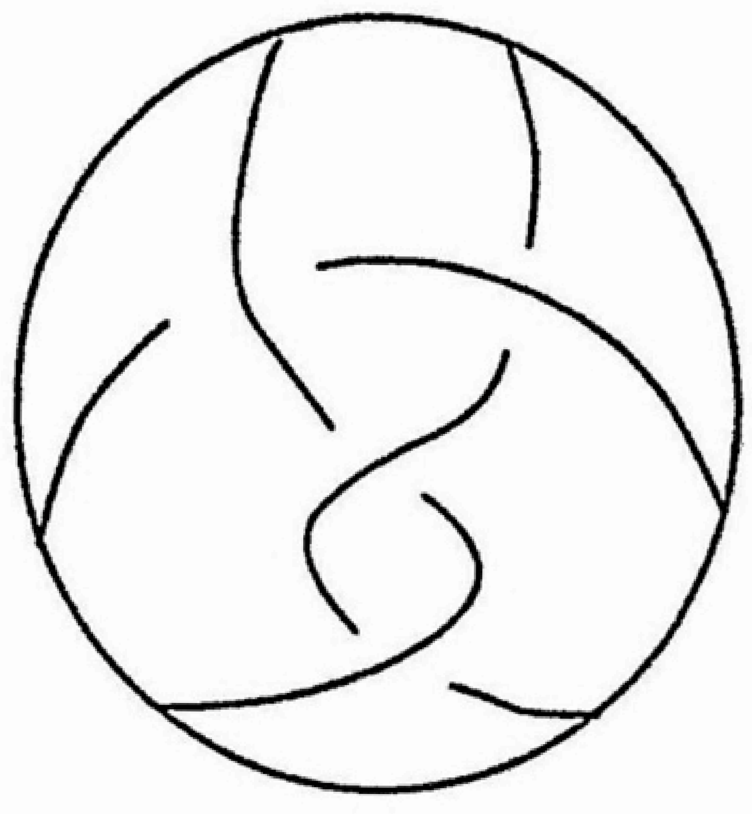
An n-component link is the image of a smooth embedding of disjoint union of n circles into S^3. In particular, a 1-component link is a knot. Let L_1 and L_2 be links in S^3. These links are *equivalent* if there is an orientation preserving piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = (L_2)$. We will describe link invariants under this equivalence. From now on, we deal with links by depicting them in a plane, then we call them *link diagrams.* Let L_1 and L_2 be two links, and D_1 and D_2 diagrams of them. Then L_1 is equivalent to L_2 in S^3 if and only if D_1 is related to D_2 by a sequence of isotopies of S^2 and the three moves, which are called the *Reidemeister moves* I, II and III (in Fig.2.1). Then the set of the equivalent classes of links, which are called link types, can be identified with the quotient set of link diagrams modulus the Reidemeister moves and isotopy of S^2 .



Reidemeister move I II III

Fig. 2.1

A *tangle* is a one-manifold properly embedded in a three-ball which consists of n-arcs and several loops. An n-tangle is a tangle with $2n$ boundary points. Tangles are considered up to ambient isotopy but in practice we will often use the word tangle for a tangle diagram or an actual embedding of a one-manifold (see for an example of 3-tangle in Fig.2.2).



3-tangle

Fig. 2.2

Seifert surface of an oriented link L is a compact oriented surface embedded in S^3 such that the boundary of the surface is equal to the link. Every link has a Seifert surface, in fact we can find a Seifert surface of a given link.

Mutation of a link is achieved by locating a tangle, with two inputs and two outputs, and flipping the tangle over, or rotating it through angle π about an axis (in such a way as to preserve the four boundary points), then the result is the other link. Mutants can thus be formed in three ways.

Now we recall the definitions of skein polynomial invariants of links L, the Jones polynomial, $V_L(t)$, and its two-variable generalization which is called Homflypt polynomial, $P_L(v, z)$.

Definition 2.1 [10, 33] The *Homflypt polynomial* of oriented links can be characterized by the recursive relation (skein relation):
(i) $v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z)$,
where L_+, L_- and L_O are three oriented links that have diagrams D_+, D_- and D_O that are the same except near a single point where they are as in Fig.2.3,
and the initial condition,
(ii) $P_{T_1} = 1$ where T_1 denotes the trivial knot.

The *Jones polynomial* of oriented link is defined as $V_L(t) = P_L(t, t^{1/2} - t^{-1/2})$. The *Alexander polynomial* $\Delta_L(t)$, as normalized by Conway, satisfies $\Delta_L(t) = P_L(1, t^{1/2} - t^{-1/2})$.

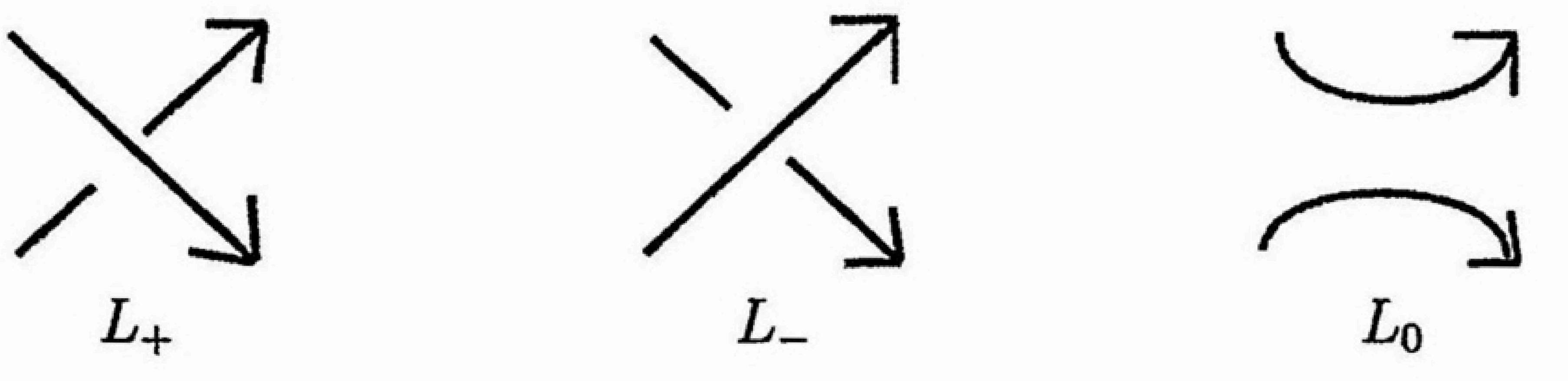


Fig. 2.3

Let F_L be a Seifert surface of an oriented link L. Denote by $\psi : H_1(F_L;\mathbb{Z}) \times H_1(F_L;\mathbb{Z}) \to \mathbb{Z}$ the Seifert form associated with F_L, i.e. $\psi(x,y) = lk(x^+, y)$, where x^+ denotes a curve pushed x slightly off F_L into the positive direction. Choosing an ordered basis of $H_1(F_L;\mathbb{Z})$ allows us to describe the Seifert form by the Seifert matrix. Let $\mathcal{A}_L$ be the Seifert matrix of the form with respect to a basis of $H_1(F_L;\mathbb{Z})$.

Let us recall the definition of the Murasugi signature, and as its generalization, Tristram-Levine signature of an oriented link.

Definition 2.2 [29, 25, 39] Let L be an oriented link in S^3 and $\mathcal{A}_L$ a Seifert matrix for L.

(i) The *Murasugi signature* of L denoted by $\sigma(L)$ is the signature of the symmetric matrix $\mathcal{A}_L + \mathcal{A}_L^t$.

Let ω be a complex number with $|\, \omega \,| = 1$ and $\omega \neq 1$, then,

(ii) The *Tristram-Levine signature* of L denoted by $\sigma_\omega(L)$ is the signature of the Hermitian matrix $(1-\omega)\mathcal{A}_L + (1-\bar{\omega})\mathcal{A}_L^t$.

The Murasugi signature is the case of $\omega = -1$ of ω-signature. The ω-signature $\sigma_\omega(L)$ is well-defined as an invariant of an oriented link L.

3 Rotation and signature invariant

Rotation was introduced as a generalization of mutation in 1987 in [3]. The idea of rotation was studied and generalized in several papers including [17], [18], [20], [32] and [35]. In this section we show that the Tristram-Levine signature is preserved by orientation-preserving rotation. Moreover, we show that any link invariant obtained from the characteristic polynomial of the Goeritz matrix, including Murasugi signature, is not changed by any rotation. In 1991, Jin and Rolfsen gave a question : Does the rotation preserve the Alexander polynomial? In 2001, P.Traczyk showed that the Conway polynomials of any pair of orientation-preserving rotants coincide. But it was still inconclusive if any orientation-reversing rotations preserve the Conway polynomials. We show that there is a pair of orientation-reversing rotants which do not share the same Alexander polynomials. This gives a negative answer for the question given by Jin and Rolfsen.

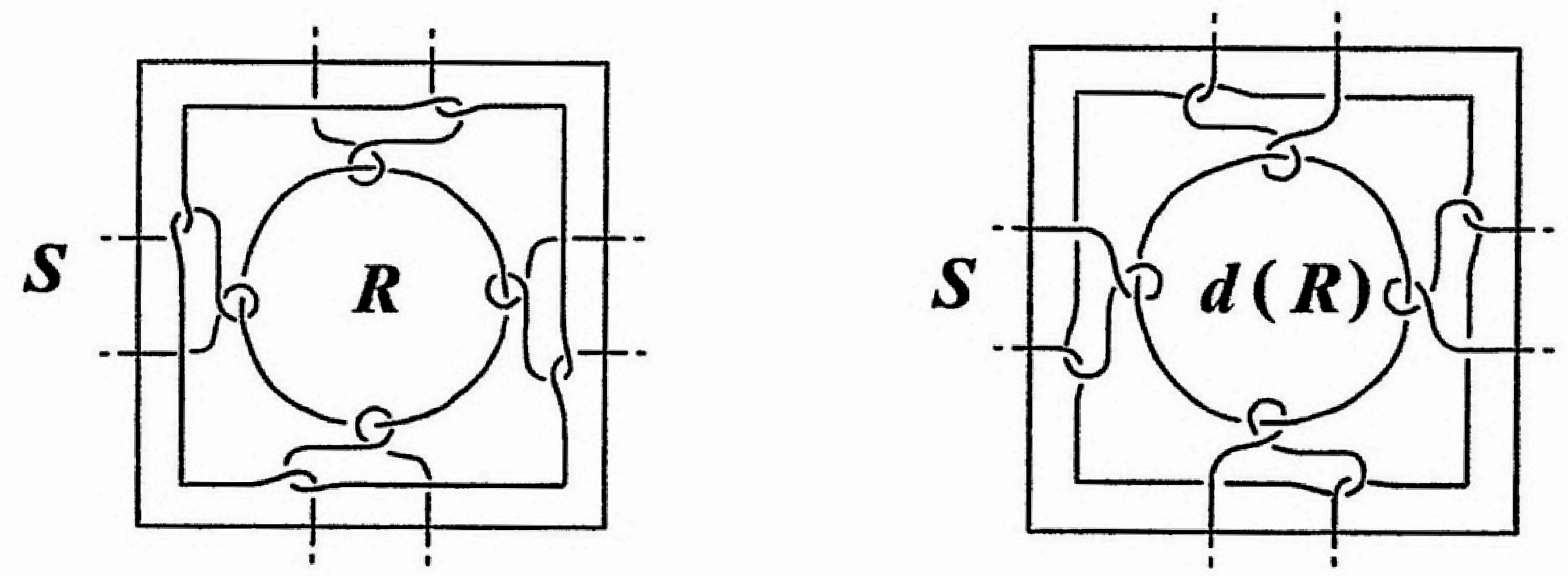
3.1 Rotation

Rotors were introduced in graph theory by Tutte [5]. The concept was adapted to knot theory in [3] as a generalization of the Conway mutation.

Consider a link L in S^3 decomposed into two n-tangles S and R (see Fig.3.1), where n-tangle ($n > 2$) is an one manifold properly embedded in a three-ball which consists of n-arcs and several loops. Let ϕ be a rotation of B^3 by the angle $\frac{2\pi}{n}$. Assume that R, called the *rotor part* of L, satisfies $\phi(R) = R$. The other tangle part S of L, is called the *stator*. Then L admits a projection decomposed into projections of S and R (also denoted by S and R) such that R lies in the regular n-gon and cuts its boundary in $2n$ points, and $\phi(R) = R$ (see Fig.3.1).

The regular n-gon has a dihedral group of symmetry D_{2n}. This group is generated by the rotation ϕ and dihedral flype d_0 which corresponds to the rotation by π along the y axis. We have the group presentation $D_{2n} = \{\phi, d_0 \mid \phi^n = {d_0}^2 = 1, d_0\phi d_0 = \phi^{-1}\}$. Set $d_{\frac{k}{2}} = \phi^k d_0$. Note that $d_{\frac{k}{2}}$ is the dihedral flype along the axis obtained from the y axis by rotating it by the angle $\frac{2\pi k}{2n}$.

A *rotant* of a link L_1 is the link L_2 (Fig.3.1 and Fig.3.3) obtained from L_1 by a dihedral flype of the rotor part. Note that L_2 is independent of the choice of a dihedral flype. We say that L_2 is obtained from L_1 by a *rotation* and L_1 and L_2 are called *a pair of rotant.*



L_1 L_2

Fig.3.1

If a link has an additional structure such as an orientation or black board framing, we assume that rotation preserves the structure. In the oriented case we allow the global change of orientation of the rotor part. More precisely for oriented rotors we have two basic choices of orientations of the boundary of a rotor : inputs and outputs alternate as in Fig.3.2(a), say *orientation-preserving rotor*, or we have pattern in-in, out-out, $\cdots$, in-in, out-out for an even n as in Fig.3.2(b), say *orientation-reversing rotor.* For oriented rotor R of an oriented link L and a dihedral flype d, the orientations of $d(R)$ and the stator parts are not compatible sometime. In this case, by reversing the orientation $d(R)$, we get an oriented link $L_2 = d(R) \cup S$ from L_1 by an oriented rotor.

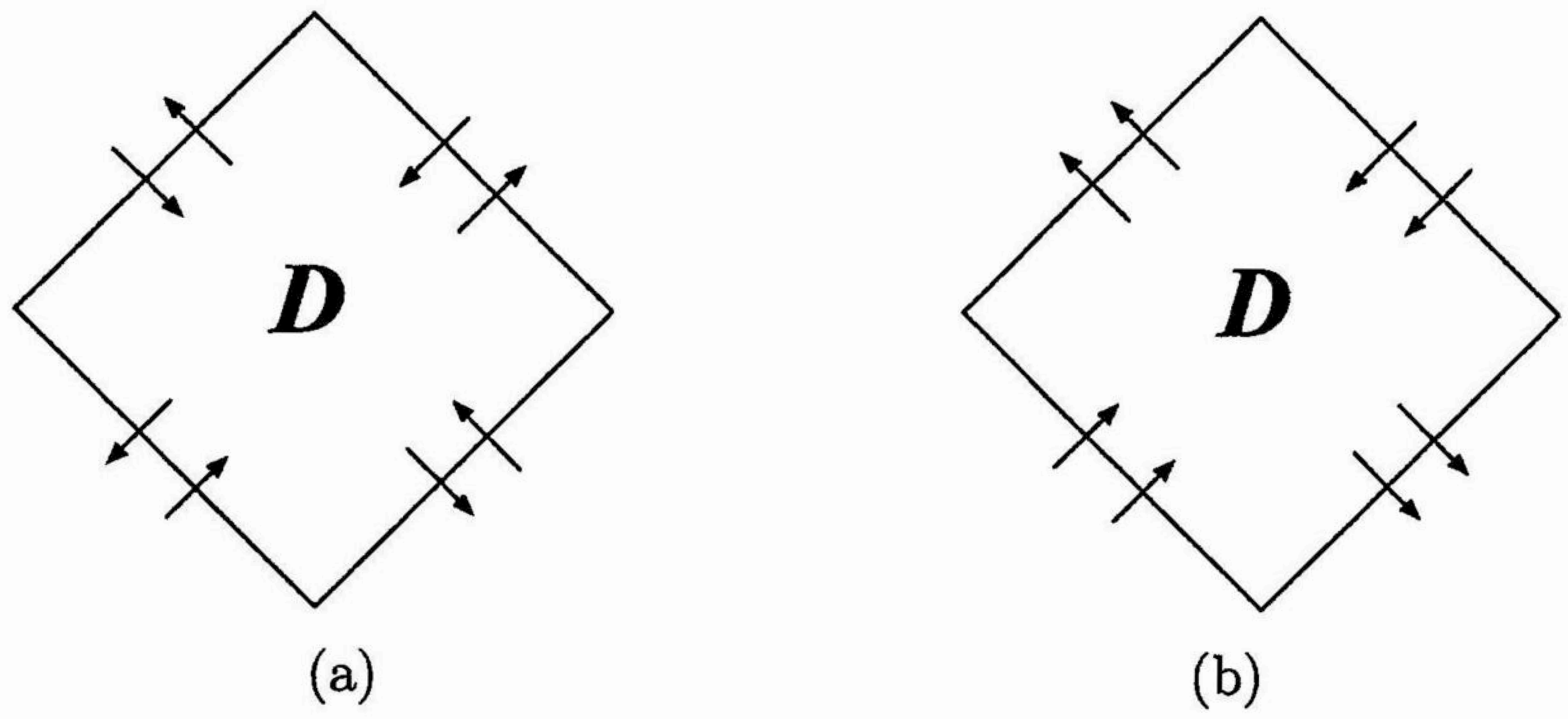


Fig.3.2

In general, it is not true that a rotation preserves the first homology of the double branched cover $M^{(2)}(L)$ of S^3 branched along L. Necessary conditions for preservation of the homology are given in [9]. Fig.3.3 ([9]) shows a pair of rotants with different $H_1(M_L^{(2)}, \mathbb{Z})$. For the link L_1 in Fig.3.3(a), $H_1(M_{L_1}^{(2)}; \mathbb{Z}) = \mathbb{Z}_{15} \oplus \mathbb{Z}_{30}$ for its orientation preserving rotant L_2 in Fig.3.3(b), we get $H_1(M_{L_2}; \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_{150}$.

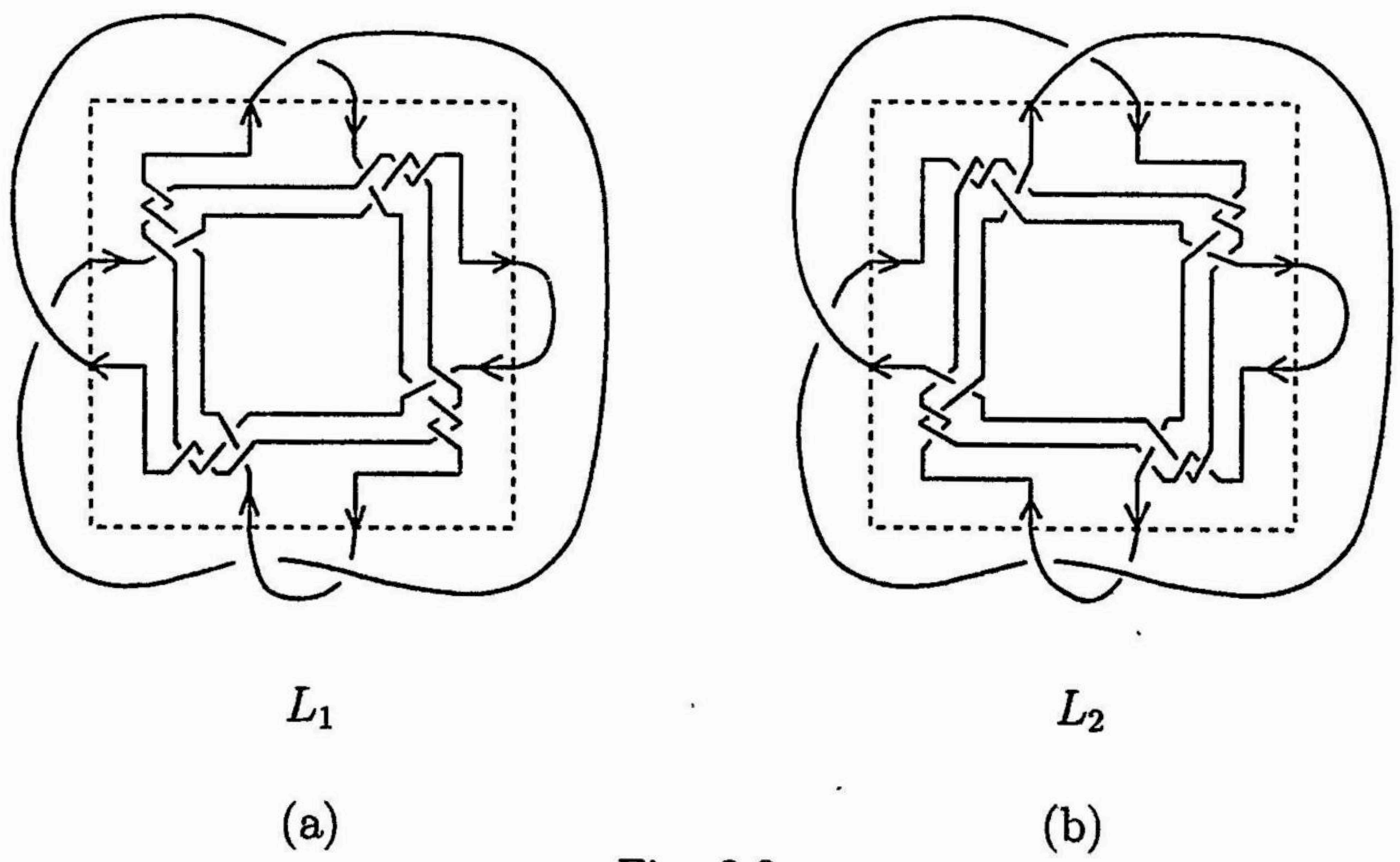


Fig. 3.3

Here $H_1(M^{(2)}_{L_k};\mathbb{Z}),(k=1,2)$ are computed by KNOT made by K.Kodama [24]. However if we assume that a pair of oriented rotants has "special" disk-band form, then the first homology groups of the double branched covers of S^3 branched along a pair of rotants are isotopic (see Theorem 4.4).

3.2 Basic properties of rotors

For an oriented link L of k-components $K_1,\cdots,K_k$ we form a linking matrix A_L with entries $a_{ij}=lk(K_i,K_j)$, where $i\neq j$. We put $a_{ii}=0$ unless L has a framing, which is an integer assigned to a component. If L has a framing then we define a_{ii} to be the framing of the ith component K_i of L. The linking matrix A_L, up to the ordering of components of L, is a link invariant. One half of the sum $\sum_{i<j}a_{ij}$ of entries of A_L outside the diagonal is the total linking number of L, denoted by $\ell k(L)$. The trace of A_L for a framed link L is denoted by $tr(L)$. Note that $tr(L)$ does not depend on the orientation of L, so $tr(L)$ is an invariant of unoriented framed link L. The following theorem describes basic properties of rotors.

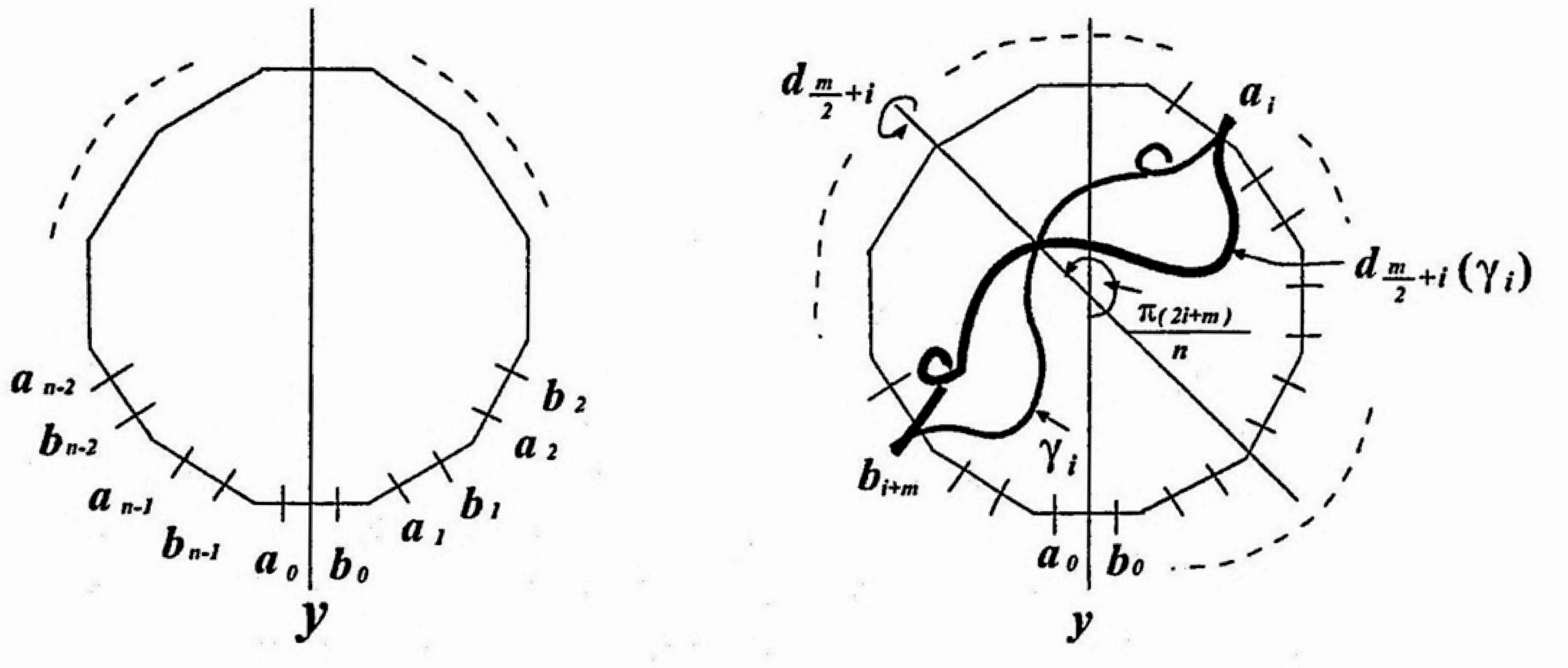
Theorem 3.1 (i) *Any rotation preserves the number of components of a link.*

(ii) *If two oriented links are related by a rotation of an oriented rotor, then the total linking number are the same.*

(iii) *If two oriented framed links are related by a rotation of an oriented rotor, and the rotor part has no closed components, then their linking matrices are the same.*

(iv) *If L is an unoriented framed link, then $tr(L)$ is preserved by any rotation.*

Proof Let R be an unoriented rotor with boundary points $a_0,b_0,a_1,b_1,\ldots,a_{n-1},b_{n-1}$, as in Fig.3.4(a). Consider the connection of a_0. Initially, we have two options: a_0 connects to either a_m or b_m for some m.



(a) (b)

Fig.3.4

The first option cannot happen for $n > 2$. Assume, by contradiction, that a_0 connects to a_m then $\phi^m(a_0) = a_m$ connects to $\phi^m(a_m) = a_{2m}$ which is the same as a_0. Therefore $2m = n$. This implies that a_i connects to $a_{i+\frac{n}{2}}$ and b_i to $b_{i+\frac{n}{2}}$. The arc $\gamma(x_i)$ of R connecting x_i to $x_{i+\frac{n}{2}}$, where the symbol x may stand for a or b, is setwise preserved by the rotation $\phi^{\frac{n}{2}}$. Therefore the arc $\gamma(x_i)$ has one fixed point, namely the point of the intersection with the z-axis. For $n > 2$, we have at least two arcs of the type $\gamma(a_i)$. Such an arc cuts the z-axis at different heights, say h_i. On the other hand $\phi(\gamma(a_i)) = \gamma(a_{i+1})$ so $h_i = h_{i+1}$, which gives a contradiction. So in this case, we have $n = 2$, and it is not hard to show the theorem.

Suppose a_0 is connected to b_m for some m. Let γ_i denote the arc connecting the point a_i with b_{i+m} in R. Consider the dihedral flype $d_{\frac{m}{2}+i}$ exchanging a_i with b_{i+m}. The image $d_{\frac{m}{2}+i}(\gamma_i)$ connects the same points on the boundary as γ_i, that is, a_i and b_{i+m} (see Fig.3.4(b)), so two boundary points of R are connected in R if and only if they are connected in $d_0(R) = d_{\frac{m}{2}+i}$. In particular, the link $L_1 = S \cup R$ and its rotant $L_2 = S \cup d_0(R)$ have the same number of components.

By the observations similar to the above, we have

Claim 3.1 (*i*) For an unoriented rotor R choose any orientation (directions) of its arcs (e.g. from a_j to b_{j+m}). Let $I(\gamma_j, \gamma_k)$ denote the sum of signs of crossings γ_j and γ_k, possibly $j = k$, then $I(\gamma_j, \gamma_k) = I(d_{\frac{j+k+m}{2}}(\gamma_j), d_{\frac{j+k+m}{2}}(\gamma_k))$.

(*ii*) For an oriented rotor R and a closed component α of R,

$I(\gamma_i, \alpha) = I(d_{\frac{m}{2}+i}(\gamma_i), d_{\frac{m}{2}+i}(\alpha))$.

Notice that $\partial\gamma_j = \partial(d_{\frac{j+k+m}{2}}(\gamma_k)), \partial\gamma_k = \partial(d_{\frac{j+k+m}{2}}(\gamma_j))$ *and* $\partial\gamma_i = \partial(d_{\frac{m}{2}+i}(\gamma_i))$.

Proof (i) The dihedral flype $d_{\frac{j+k+m}{2}}$ of R sends a_j to b_{k+m} and a_k to b_{j+m}, thus it sends the arc γ_j, connecting a_j to b_{j+m} in R (resp. a_k to b_{k+m}) to the arc $d_{\frac{j+k+m}{2}}(\gamma_k)$ connecting b_{k+m} to a_k in $d_0(R)$ (resp. $d_{\frac{j+k+m}{2}}(\gamma_j)$ connecting b_{j+m} to a_k) (see Fig.3.5). Therefore $I(\gamma_j, \gamma_k) = I(d_{\frac{j+k+m}{2}}(\gamma_k), d_{\frac{j+k+m}{2}}(\gamma_j))$, as required.

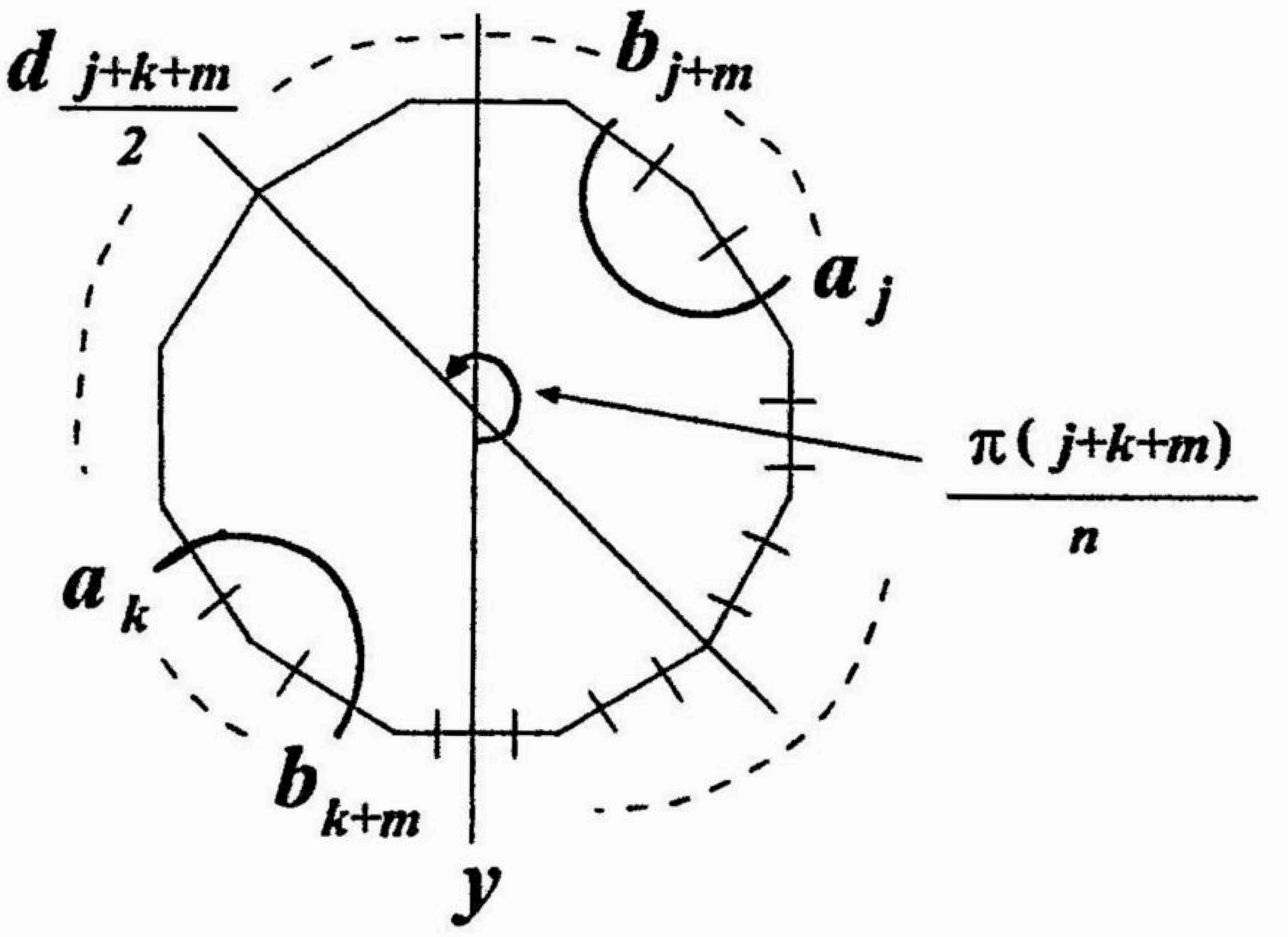


Fig. 3.5

(ii) Since γ_i in R and $d_{\frac{m}{2}+i}(\gamma_i)$ in $d_0(R)$ connect the same boundary points a_i and b_{i+m}, we have the conclusion.

Theorem 3.1 (ii), (iii) and (iv) follow from Claim 3.1 and the fact that L_1 and L_2 have the same stator. □

3.3 Unoriented rotation and Murasugi signature

In this section we prove that the Murasugi signature of unoriented links [13, 29, 30] is preserved by any rotation. We place this result in a more general setting by proving that the rotation preserves the characteristic polynomial of the Goeritz matrix (with the specific choices of surfaces). This allows us to obtain the result mentioned in [32] which was also proven by Traczyk that the determinant of an unoriented link is preserved by any rotation.

Let F_L be a spanning surface, possibly nonorientable, of an unoriented link L. We use the following generalization of Seifert and Goeritz forms described by C.M.Gordon and R.A.Litherland in [13]. For the spanning surface F_L, consider the I-bundle $N(F_L)$ over F_L. Then the ∂I-bundle $\tilde{F_L}$ is a double cover of F_L with the projection map $p : \tilde{F_L} \longrightarrow F_L$. The bilinear form $\mathcal{G}_{F_L} : H_1(F_L;\mathbb{Z}) \times H_1(F_L;\mathbb{Z}) \longrightarrow \mathbb{Z}$ defined by $\mathcal{G}_{F_L}(x,y) = lk(p^{-1}x, y)$, where x and y are oriented loops in F_L, is called the *Goeritz form* associated to the surface F_L. For an ordered basis of $H_1(F_L;\mathbb{Z})$ we have the matrix G_{F_L} describing the form $\mathcal{G}_{F_L}$. The matrix G_{F_L} is called the *Goeritz matrix* of F_L with respect to a basis of $H_1(F_L;\mathbb{Z})$.

Let F_L be an unoriented spanning surface of a link L. For an ordered basis of $H_1(F_L;\mathbb{Z})$ we have the Goeritz matrix G_{F_L} representing the Goeritz form $\mathcal{G}_{F_L}$. The form $\mathcal{G}_{F_L}$ defined over $\mathbb{Z}$ can be extended to the form $\hat{\mathcal{G}}_{F_L}$ over $\mathbb{C}$. We view the form $\hat{\mathcal{G}}_{F_L}$ as Hermitian form represented in a basis by the Hermitian matrix $\hat{G}_{F_L}$ (i.e. $\hat{G}_{F_L} = \overline{\hat{G}^t_{F_L}}$).

For a spanning surface F_{L_k} of $L_k = K_{k1} \cup K_{k2} \cup \cdots \cup K_{km}$, the framing of L_k is uniquely defined from F_{L_k} as follows : Let ${K_{ki}}^{F_{L_k}}$ be a parallel copy of K_{ki} missing F_{L_k}, then we define the framing K_{ki} to be $lk(K_{ki}, {K_{ki}}^{F_{L_k}})$. We put $e(F_{L_k}) = -\sum_i lk(K_{ki}, {K_{ki}}^{F_{L_k}}) = -tr(L_k)$.

Definition 3.1 [13, 30] Let L be an unoriented link in S^3, and let $\hat{L}$ be the link obtained from L by a choice of an orientation. The *Murasugi signature* $\hat{\sigma}(L)$ of an unoriented link L is defined to be $\hat{\sigma}(L) = \sigma(\hat{L}) + \ell k(\hat{L})$.

Remark 3.1 *Murasugi showed in [30] that* $\sigma(\hat{L})+\ell k(\hat{L})$ *is independent of the choice of an orientation of* L. *So* $\hat{\sigma}(L)$ *is an invariant of unoriented links. We shall use later the fact that* $\hat{\sigma}(L) = sign(G_{F_L}) + \frac{1}{2}e(F_L)$ *[13].*

Murasugi signature of unoriented links is preserved by any rotation. In fact we have :

Theorem 3.2 *Let* L_1 *and* L_2 *be a pair of unoriented rotants of any order* n. *Then* $\hat{\sigma}(L_1) = \hat{\sigma}(L_2)$.

Let L_1 and L_2 be a pair of unoriented rotant links. Consider projections of the links L_1 and L_2 onto $\mathbb{R}^2$ with rotor parts R_1 and R_2 contained in the disks D_1 and D_2, respectively. We can deform the stator parts S_1 and S_2 of the diagrams of L_1 and L_2 into the position shown in Fig.3.6.

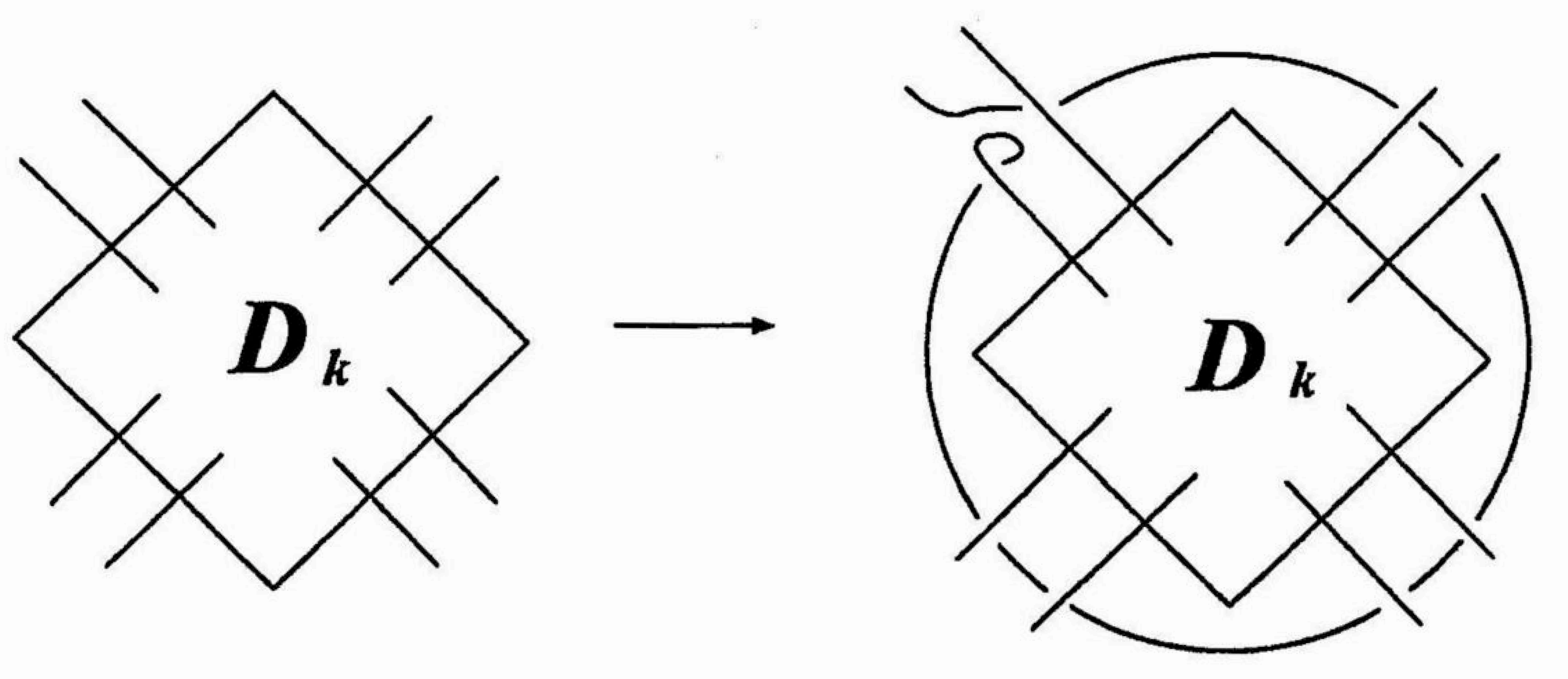


Fig.3.6

We color the regions on $\mathbb{R}^2$ bounded by the diagrams of L_k in checkerboard manner as in Fig.3.7. We may assume that the unbounded region is colored in white. Using the black regions we form the spanning surface F_{L_k} for $k = 1, 2$. We choose the boundary curves of the white bounded regions with the anti-clockwise orientations as a basis of $H_1(F_{L_k}; \mathbb{Z})$. We call them the *standard basis.*

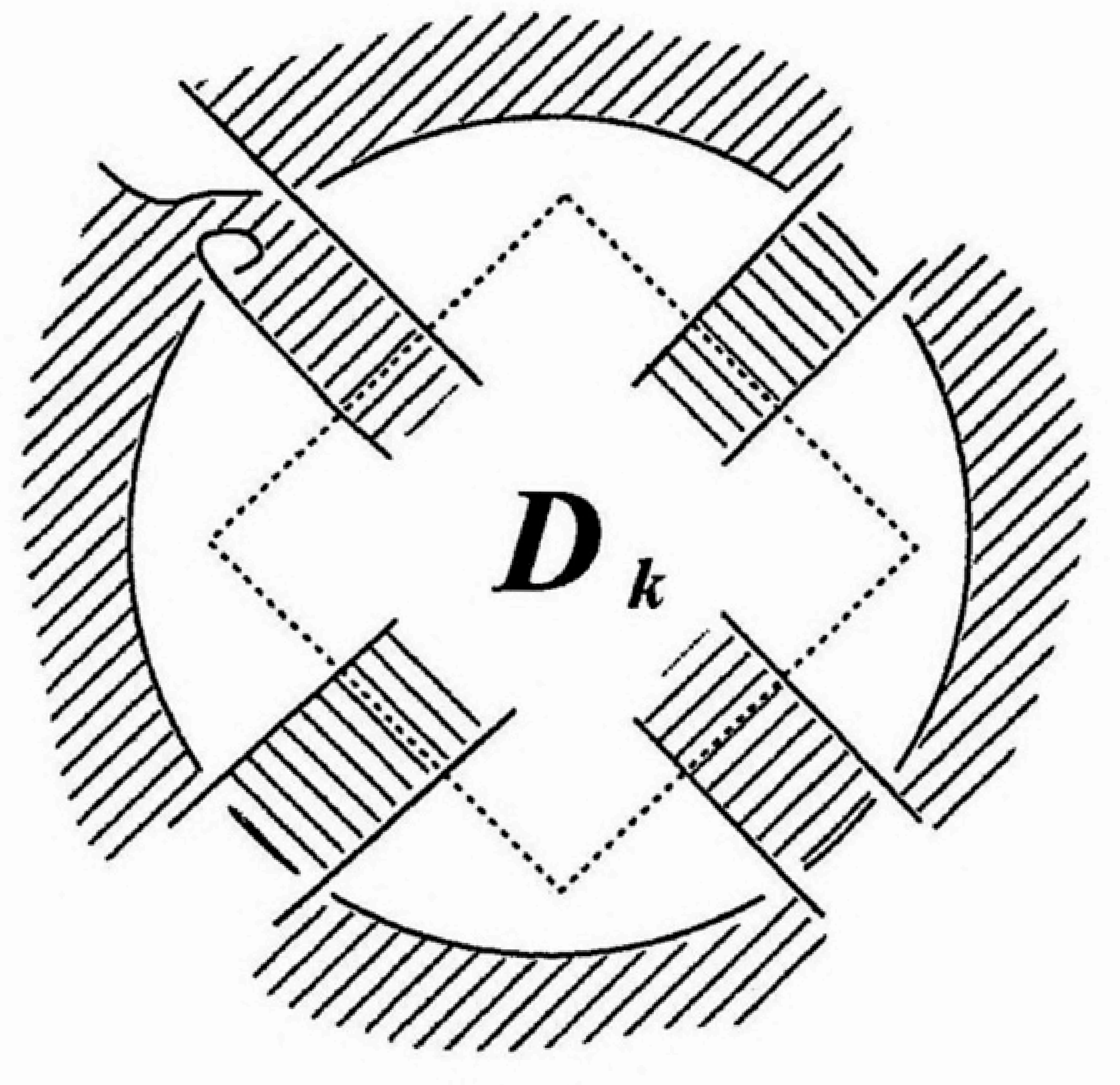


Fig.3.7

We may assume that the framed links L_1 and L_2 obtained from F_1 and F_2 respectively are a pair of rotants. By Theorem 3.1, $tr(L_1) = tr(L_2)$, so we have $e(F_{L_1}) = e(F_{L_2})$. This fact, Remark 3.1 and the following theorem imply Theorem 3.2.

With the choices made above, we can formulate the main result of this section.

Theorem 3.3 *Let G_{F_k} $(k = 1, 2)$ be the Goeritz matrices with respect to the standard basis. Then* $\det(G_{F_{L_1}} - \lambda E) = \det(G_{F_{L_2}} - \lambda E)$.

Proof Let $X_{\mathcal{S}_k}$ and $X_{\mathcal{R}_k}$ be the subsets of the standard basis of $H_1(F_{L_k}; \mathbb{Z})$ which "live entirely" in the stator and rotor part respectively, and let $X_{\mathcal{M}_k}$ be the complement of $X_{\mathcal{S}_k} \cup X_{\mathcal{R}_k}$. Consider submodules $\mathcal{S}_k$, $\mathcal{R}_k$ and $\mathcal{M}_k$ of $H_1(F_{L_k}; \mathbb{Z})$ generated by $X_{\mathcal{S}_k}$, $X_{\mathcal{R}_k}$ and $X_{\mathcal{M}_k}$. We have the following decomposition into the direct sum of $\mathbb{Z}$-modules : $H_1(F_{L_k}; \mathbb{Z}) = \mathcal{S}_k \oplus \mathcal{M}_k \oplus \mathcal{R}_k$. Let v denote the generator of $\mathcal{M}_1$ intersecting the axis y of the dihedral flype d (see Fig.3.8).

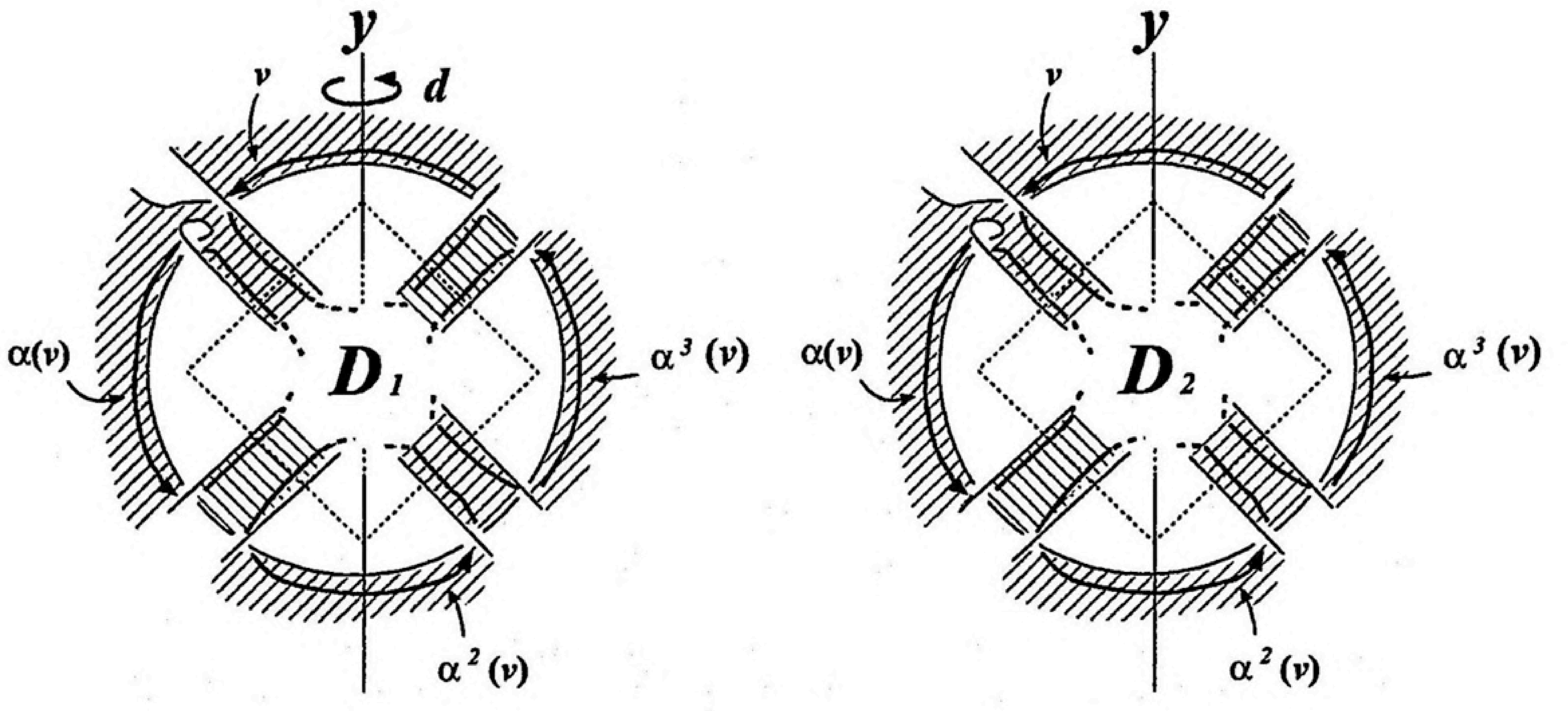


Fig.3.8

There is an action of the cyclic group $\mathbb{Z}_n =< \alpha >$ on $\mathcal{R}_1 \oplus \mathcal{M}_1$ induced by the $\frac{2\pi}{n}$-rotation around the center of D_1. Thus the ordered set $X_{\mathcal{M}_1} : v, \alpha(v), \alpha^2(v), \cdots, \alpha^{n-1}(v)$ is a basis of $\mathcal{M}_1$. Let $X^*_{\mathcal{R}_1}$ be a set of generators of $\mathcal{R}_1$ formed by choosing one representative from each orbit of $\mathbb{Z}_n$-action on standard generators of $\mathcal{R}_1$ (i.e. $X^*_{\mathcal{R}_1} = X_{\mathcal{R}_1}/\mathbb{Z}_n$). We construct the bijection η on the set of standard generators of $H_1(F_{L_k}; \mathbb{Z})$. First, we put $\eta|_{X_{\mathcal{S}_1}} : X_{\mathcal{S}_1} \to X_{\mathcal{S}_2}$ to be the identity map because the stator part is unchanged. The map $\eta|_{X_{\mathcal{M}_1}} : X_{\mathcal{M}_1} \to X_{\mathcal{M}_2}$ is given by $\eta(\alpha^j(v)) = \alpha^j(d(v))$ (i.e. $\alpha^j(v)$ and $\eta(\alpha^j(v))$ have the same stator parts). Finally, $\eta|_{X_{\mathcal{R}_1}} : X_{\mathcal{R}_1} \to X_{\mathcal{R}_2}$ is given by $\eta(\alpha^j(x)) = d(\alpha^j(x))$ for $x \in X_{\mathcal{R}_1}$. The bijection η extends to the isomorphism, $H_1(F_{L_1}; \mathbb{Z}) \to H_1(F_{L_2}; \mathbb{Z})$, is also denoted by η, since it is easily extendable to each component of the direct sum of $H_1(F_{L_1}; \mathbb{Z})$. We use the isomorphism η to identify $H_1(F_{L_1}; \mathbb{Z})$ with $H_1(F_{L_2}; \mathbb{Z})$. This identification allows us to drop indices of $\mathcal{S}_k, \mathcal{M}_k$ and $\mathcal{R}_k$ and write $\mathcal{S}, \mathcal{M}$ and $\mathcal{R}$.

Let us consider forms $\mathcal{G}_1 = \mathcal{G}_{F_{L_1}}$ and $\mathcal{G}_2 = \mathcal{G}_{F_{L_2}}$ on the same space $\mathcal{S} \oplus \mathcal{M} \oplus \mathcal{R}$. We have the following relations for $\mathcal{G}_1$ and $\mathcal{G}_2$.

(1) $\mathcal{G}_2(x,y) = \mathcal{G}_1(x,y)$ for all $x, y \in \mathcal{S} \oplus \mathcal{M}$.

(2) $\mathcal{G}_2(x,y) = \mathcal{G}_1(x,y)$ for all generators $x, y \in \mathcal{R}$.

(3) $\mathcal{G}_1(x,y) = \mathcal{G}_1(\alpha^l(x), \alpha^l(y))$ for all generators $x, y \in \mathcal{M} \oplus \mathcal{R}$,

$\mathcal{G}_2(x, \alpha^l(v)) = \mathcal{G}_1(x, \alpha^{-l}(v))$ for every generator x of $\mathcal{R}$,

$\mathcal{G}_2(\alpha^l(x), v) = \mathcal{G}_1(\alpha^l(x), v)$ for every generator x of $\mathcal{R}$, and

$\mathcal{G}_2(x, v) = \mathcal{G}_2(\alpha^l(x), \alpha^{-l}(v))$ for every generator $x \in \mathcal{R}$.

(4) $\mathcal{G}_k(x,y) = 0$ for all $x \in \mathcal{S}', y \in \mathcal{R}(k = 1,2)$.

Let $\mathbf{S}, \mathbf{M}$ and $\mathbf{R}$ be the subspaces of $(\mathcal{S} \oplus \mathcal{M} \oplus \mathcal{R}) \otimes \mathbb{C}$ complexifications of $\mathcal{S}, \mathcal{M}$ and $\mathcal{R}$, respectively. We have the involution $^- : \mathbf{S} \oplus \mathbf{M} \oplus \mathbf{R} \to \mathbf{S} \oplus \mathbf{M} \oplus \mathbf{R}$ corresponding to the conjugation in the factor $\mathbb{C}$ of the tensor product. The image of $x \in \mathbf{S} \oplus \mathbf{M} \oplus \mathbf{R}$ under this involution is denoted by $\bar{x}$. Using the rotational symmetry of the rotor part we change "conveniently" generating sets of $\mathbf{M}$ and $\mathbf{R}$ in the following way. Let ω_j be an n-th root of unity, $\omega_j = e^{2\pi i \frac{j}{n}}$. We replace the generating set $\{\alpha^j(v) \mid j = 0, 1, \cdots, n-1\}$ of $\mathbf{M}$ by $\{\mathbf{v}_j \mid \mathbf{v}_j = \sum_{l=0}^{n-1} \omega_j^l \alpha^l(v), j = 0, 1, \cdots, n-1\}$. For $\mathbf{R}$ we consider two choices of generating sets related by involution $^-$. We replace $\{\alpha^j(y_p) \mid y_p \in X^*_{\mathcal{R}}, j = 0, 1, \cdots, n-1\}$, by $\{\mathbf{y}_{j,p} \mid \mathbf{y}_{j,p} = \sum_{l=0}^{n-1} \omega_j^l \alpha^l(y_p), y_p \in X^*_{\mathcal{R}}, j = 0, 1, \cdots, n-1\}$ or $\{\overline{\mathbf{y}_{j,p}} \mid \overline{\mathbf{y}_{j,p}} = \sum_{l=0}^{n-1} \overline{\omega_j}^l \alpha^l(y_p), y_p \in X^*_{\mathcal{R}}, j = 0, 1, \cdots, n-1\}$.

Let us consider forms $\hat{\mathcal{G}}_1 = \hat{\mathcal{G}}_{F_{L_1}}$ and $\hat{\mathcal{G}}_2 = \hat{\mathcal{G}}_{F_{L_2}}$ on the same space $\mathbf{S} \oplus \mathbf{M} \oplus \mathbf{R}$.

This new generating sets for $\mathbf{M} \oplus \mathbf{R}$ satisfy the following conditions.

(1) $\hat{\mathcal{G}}_k(\mathbf{v}_j, \mathbf{v}_m) = 0$ for $j \neq m$, where $\mathbf{v}_j, \mathbf{v}_m \in \mathbf{M}$ and $k = 1, 2$,

$\hat{\mathcal{G}}_1(\mathbf{x}_{j,p}, \mathbf{v}_m) = \hat{\mathcal{G}}_2(\overline{\mathbf{x}_{j,p}}, \mathbf{v}_m) = 0$ for $j \neq m$ where $\mathbf{x}_{j,p} \in \mathbf{R}_1, \overline{\mathbf{x}_{j,p}} \in \mathbf{R}_2, \mathbf{v}_m \in \mathbf{M}$,

$\hat{\mathcal{G}}_1(\mathbf{x}_{j,p}, \mathbf{y}_{m,q}) = \hat{\mathcal{G}}_2(\overline{\mathbf{x}_{j,p}}, \overline{\mathbf{y}_{m,q}})$ for $j \neq m$ where $\mathbf{x}_{j,p}, \mathbf{y}_{m,q} \in \mathbf{R}_1, \overline{\mathbf{x}_{j,p}}, \overline{\mathbf{y}_{m,q}} \in \mathbf{R}_2$.

(2) $\hat{\mathcal{G}}_1(\mathbf{x}, \mathbf{y}_{j,p}) = \hat{\mathcal{G}}_2(\mathbf{x}, \overline{\mathbf{y}_{j,p}}) = 0$ for any $\mathbf{x} \in \mathbf{S}, \mathbf{y}_{j,p} \in \mathbf{R}_1, \overline{\mathbf{y}_{j,p}} \in \mathbf{R}_2$.

(3) $\hat{\mathcal{G}}_1(\mathbf{y}_{j,p}, \mathbf{y}_{j,q}) = \overline{\hat{\mathcal{G}}_2(\overline{\mathbf{y}_{j,p}}, \overline{\mathbf{y}_{j,q}})}$ for any $\mathbf{y}_{j,p}, \mathbf{y}_{j,q} \in \mathbf{R}_1, \overline{\mathbf{y}_{j,p}}, \overline{\mathbf{y}_{j,q}} \in \mathbf{R}_2$.

(4) $\hat{\mathcal{G}}_1(\mathbf{v}_j, \mathbf{y}_{j,p}) = \overline{\hat{\mathcal{G}}_2(\mathbf{v}_j, \overline{\mathbf{y}_{j,p}})}$ for any $\mathbf{v}_j \in \mathbf{M}, \mathbf{y}_{j,p} \in \mathbf{R}_1, \overline{\mathbf{y}_{j,p}} \in \mathbf{R}_2$, and

$\hat{\mathcal{G}}_1(\mathbf{v}_j, \mathbf{v}_j) = \hat{\mathcal{G}}_2(\mathbf{v}_j, \mathbf{v}_j)$ for any $\mathbf{v}_j \in \mathbf{M}$.

(5) $\hat{\mathcal{G}}_1(\mathbf{x}, \mathbf{v}_j) = \hat{\mathcal{G}}_2(\mathbf{x}, \mathbf{v}_j)$ for any $\mathbf{x} \in \mathbf{S}, \mathbf{v}_j \in \mathbf{M}$.

Take the subspace W_j of $\mathbf{M} \oplus \mathbf{R}$ corresponding to ω_j, and choose its ordered basis by taking $\mathbf{v}_j$ from $\mathbf{M}$ first and $\mathbf{y}_{j,p}$ from $\mathbf{R}$ in arbitrary order. To obtain the ordered basis of $\mathbf{M} \oplus \mathbf{R}$ we place the basis of W_j before the basis of W_{j+1} for $j = 0, 1, \cdots, n-1$. Finally we add ordered basis of $\mathbf{S}$. Then we have an orderd basis of $H_1(F_L; \mathbb{C})$. We also have an ordered basis of $H_1(F_{L_2}; \mathbb{C})$ by replacing each $\mathbf{y}_{j,p}$ with $\overline{\mathbf{y}_{j,p}}$.

We obtain the matrices of forms $\hat{\mathcal{G}}_1$ and $\hat{\mathcal{G}}_2$ in ordered bases of $\mathbf{S} \oplus \mathbf{M} \oplus \mathbf{R}$ as described below.

$$\hat{G}_{L_1} = \begin{pmatrix} B_{10} & & \mathbf{0} & {}^t\overline{S_0} \\ & \ddots & & \vdots \\ \mathbf{0} & & B_{1n-1} & {}^t\overline{S_{n-1}} \\ S_0 & \cdots & S_{n-1} & S \end{pmatrix} \text{ and } \hat{G}_{L_2} = \begin{pmatrix} B_{20} & & \mathbf{0} & {}^t\overline{S_0} \\ & \ddots & & \vdots \\ \mathbf{0} & & B_{2n-1} & {}^t\overline{S_{n-1}} \\ S_0 & \cdots & S_{n-1} & S \end{pmatrix}.$$

In those bases, B_{1j} (respectively B_{2j}), where $j = 0, 1, \cdots, n-1$, is the matrix of the restriction of the form $\hat{\mathcal{G}}_1$ (and $\hat{\mathcal{G}}_2$ respectively) to the subspace W_j generated by $\{\mathbf{v}_j\} \cup \{\mathbf{y}_{j,p} \mid y_p \in X^*_{\mathcal{R}_1}\}$ ($\{\mathbf{v}_j\} \cup \{\overline{\mathbf{y}_{j,p}} \mid y_p \in X^*_{\mathcal{R}_1}\}$ respectively). Finally S is the restriction to the stator part which is the same for $\hat{\mathcal{G}}_1$ and $\hat{\mathcal{G}}_2$. Notice that ${B_{1k}}^t = B_{2k}$, $S_l = (\mathbf{s}_{l1}\ \mathbf{0} \cdots \mathbf{0})$, and $\mathbf{s}_{l1}$ is the first column of each matrix S_l.

Matrices $M_k = (\hat{G}_{L_k} - \lambda E)$ $(k = 1, 2)$ satisfy the conditions of Traczyk's Proposition 2.9 in [38] for any real number λ. Thus $\det(M_1) = \det(M_2)$ for any real λ. So $\det(M_1) = \det(M_2)$ for any complex λ as well. $\square$

3.4 Oriented rotation and Tristram-Levine signature

In this section we can generalize Traczyk's method in [38] for orientation-preserving rotors (see Fig.3.3(a)) to show that the characteristic polynomial of the Hermitian form associated with Seifert form is invariant under orientation-preserving rotation.

Theorem 3.4 *Let L_1 and L_2 be a pair of orientation-preserving rotants of an order n. Then* $\sigma_\omega(L_1) = \sigma_\omega(L_2)$.

Traczyk proved that the Alexander polynomials for the orientation-preserving rotants coincide in [38]. His proof is still valid for characteristic polynomials.

Let S^2 be the sphere of projection of the diagram of L, and F_L the Seifert surface of L. Let H be a trivalent graph that consists of the Seifert circles and the cores of the bands. Let $R_1, R_2, \cdots, R_m$ be the components of $S^2 - H$ which are not bounded by Seifert circles. Assign the anti-clockwise orientation for each boundary curves of the regions $R_i (i = 1, \cdots, m)$; then these curves are generators of $H_1(F_L; \mathbb{Z})$. Whenever we refer to generators of $H_1(F_L; \mathbb{Z})$, we mean this particular set of standard generators for Seifert surface F_L.

Let L_1 and L_2 be a pair of orientation-preserving n-rotant links. Now we deform the diagrams of L_k $(k = 1, 2)$ on S^2 into the position for which the computation is feasible. We follow Traczyk's paper [38]. Let D_k be a disk in S^2 which is the rotor part of the diagram of L_k $(k = 1, 2)$, and $\bar{D}$ $(= S^2 - \mathrm{int} D_k)$ the stator part. The disks D_k $(k = 1, 2)$ and $\bar{D}$ are all n-tangles. By deforming the stator part as in Fig.3.9, we have an outermost Seifert circle C in $\bar{D}$ that is parallel to ∂D_k. Let $\overline{D_C}$ be the region which is bounded by C and ∂D_k in $\bar{D}$. The rotational symmetries of the rotor parts D_k $(k = 1, 2)$ are extended to the part $D_k \cup \overline{D_C}$, i.e., we may assume that $D_k \cup \overline{D_c} (k = 1, 2)$ are n-rotors.

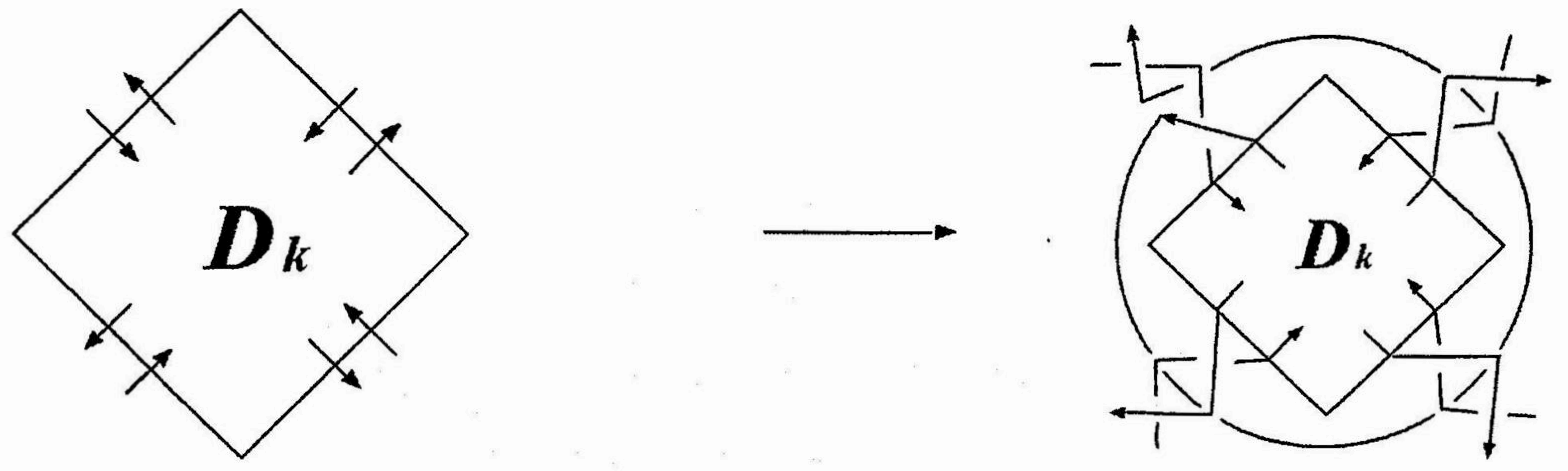


Fig. 3.9

Let F_{L_k} $(k = 1, 2)$ be Seifert surfaces for L_k (Fig.3.10). Let $\mathcal{A}_{L_k}$ be the Seifert matrix for L_k.

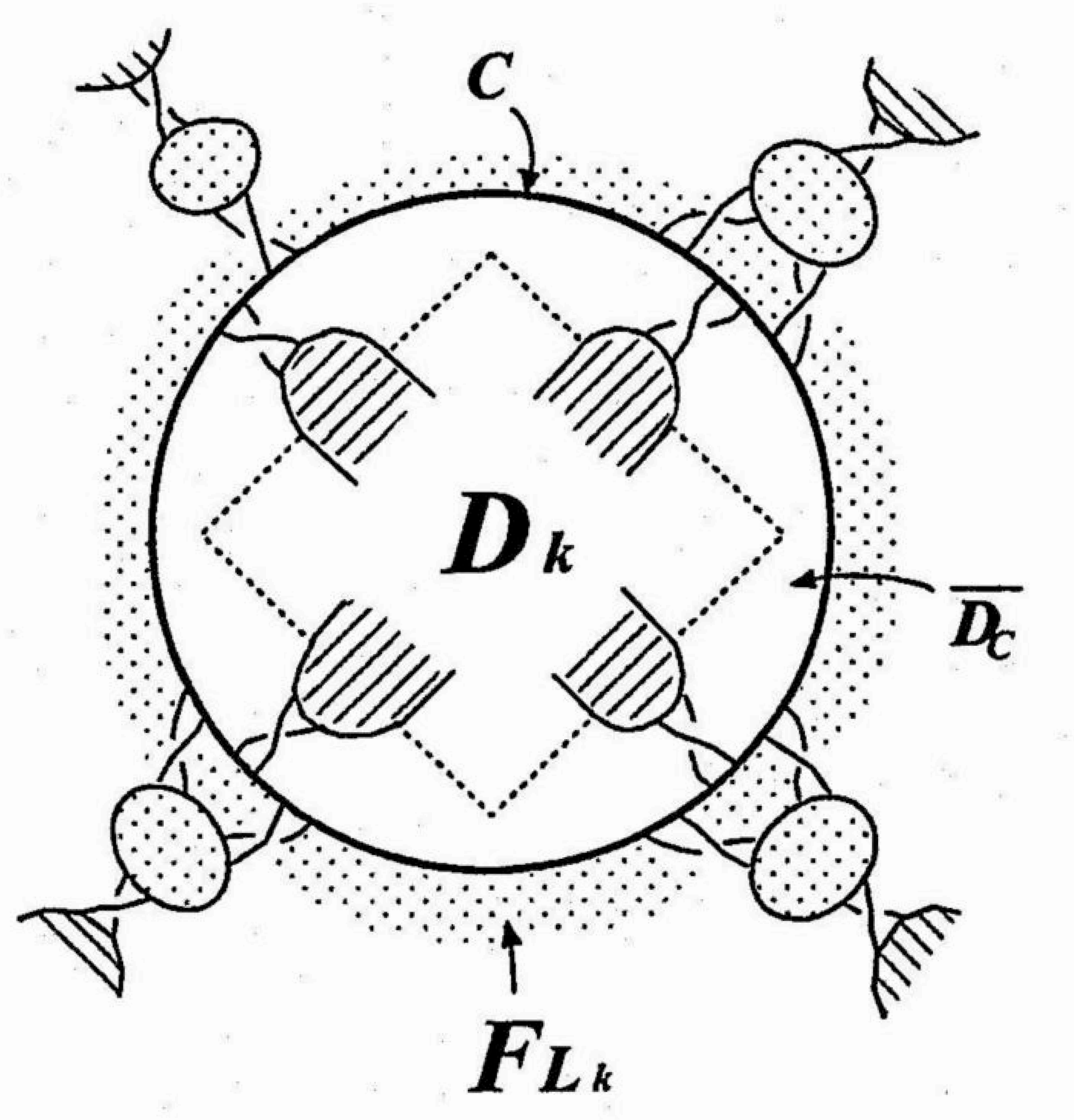


Fig. 3.10

Let ξ be a complex number and $X_{L_k} = \xi\mathcal{A}_{L_k} + \bar{\xi}{\mathcal{A}_{L_k}}^t$ a Hermitian matrix which represents an Hermitian form $\theta(x, y) = \xi\psi(x, y) + \bar{\xi}\psi(y, x)$ of $x, y \in H_1(F_{L_k}; \mathbb{Z})$.

With the choices made above, we can formulate the main result of this section.

Theorem 3.5 *The characteristic polynomials of Hermitian matrices X_{L_1} and X_{L_2} coincide.*

Proof Here, we consider three submodules $\mathcal{S}_k, \mathcal{R}_k$ and $\mathcal{M}_k$ of $H_1(F_{L_k};\mathbb{Z})$, where $\mathcal{S}_k, \mathcal{R}_k$ and $\mathcal{M}_k$ are generated by the sets $X_{\mathcal{S}_k}, X_{\mathcal{R}_k}$, and $X_{\mathcal{M}_k}$ standard generators of $H_1(F_{L_k};\mathbb{Z})$ which live entirely in the stator part $\bar{D}$, rotor part D_k, and partially in $\bar{D}$ and partially in D_k $(k = 1,2)$, respectively. Clearly we have the following splitting; $H_1(F_{L_k};\mathbb{Z}) = \mathcal{S}_k \oplus (\mathcal{M}_k + \mathcal{R}_k)$ $(k = 1,2)$.

Let v denote the generator of $\mathcal{M}_1$ intersecting with the axis of dihedral flype (see Fig.3.11). There is an action of the cyclic group $\mathbb{Z}_n =< \alpha >$ on $\mathcal{R}_1 + \mathcal{M}_1$ induced by the $\frac{2\pi}{n}$-rotation around the center of D_1.

Then, the set $X_{\mathcal{M}_k} = \{v, \alpha(v), \alpha^2(v), \cdots, \alpha^{n-1}(v)\}$ generates $\mathcal{M}_1$. Let $\alpha^j(v)$ also denote a generator of $\mathcal{M}_2$ that coincides with $\alpha^j(v)$ of $\mathcal{M}_1$ in $\overline{D_C}$. The submodule $\mathcal{R}_1$ is generated by the set $\{\alpha^j(x) \mid x \in X_{\mathcal{R}_1}, j = 0,1,\cdots,n-1\}$. Since D_2 is the image of D_1 by the dihedral flip d around the axis y which is across v, $\mathcal{R}_2$ is generated by $\{d(\alpha^j(x)) \mid x \in X_{\mathcal{R}_1}, j = 0,1,\cdots,n-1\}$ (Fig.3.11). In order to compare ψ_1 with ψ_2, we denote by $\alpha^j(x)$ the generator $d(\alpha^j(x)) \in \mathcal{R}_2$ $(j = 0,1,2,\cdots,n-1)$.

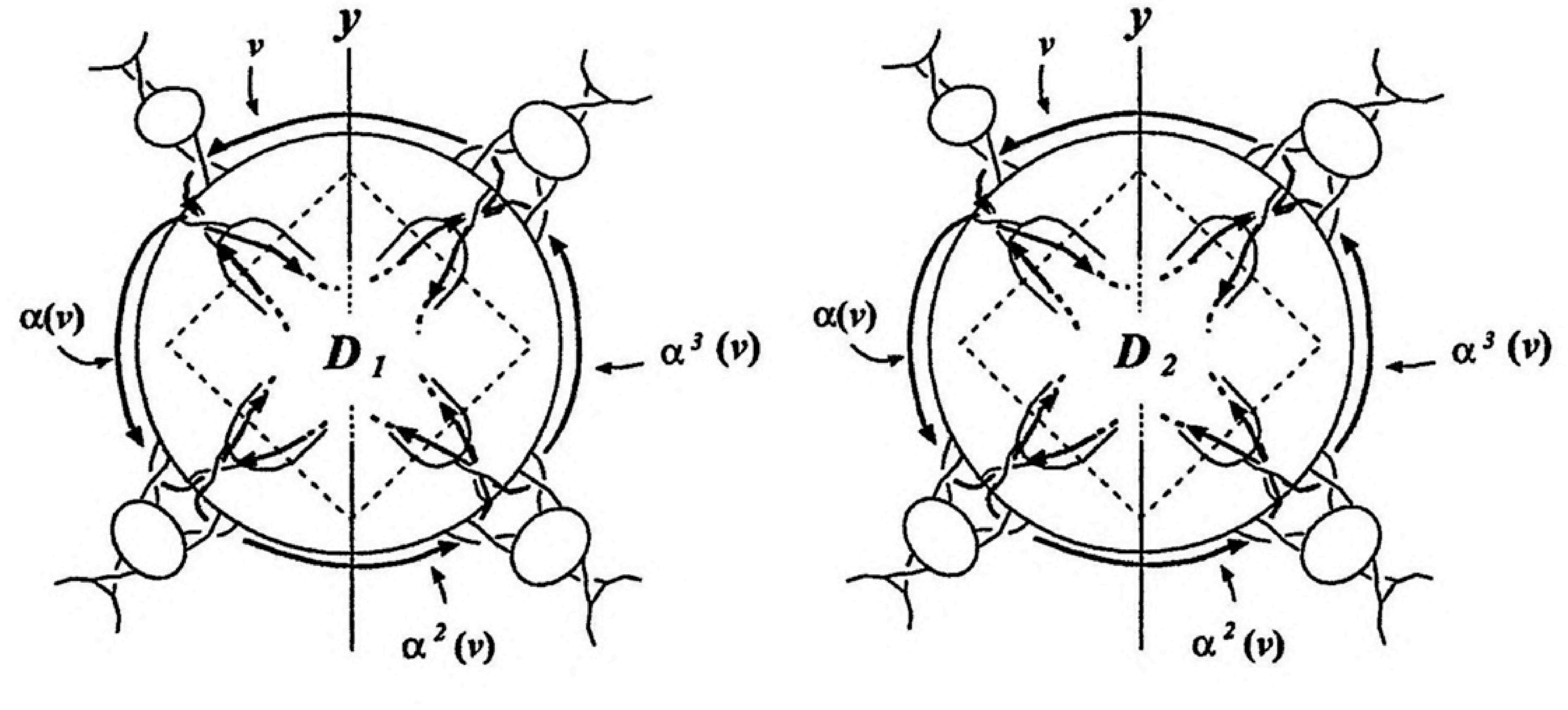


Fig.3.11

Under this identification, the relations between ψ_1 and ψ_2 are the followings. Here, we omit the indices of $\mathcal{S}, \mathcal{M}$ and $\mathcal{R}$.

(1) $\psi_2(x,y) = \psi_1(x,y)$ for all $x, y \in \mathcal{S} + \mathcal{M}$.

(2) $\psi_2(x,y) = \psi_1(y,x)$ for all $x, y \in \mathcal{R}$.

(3) $\psi_2(x, \alpha^j(v)) = \psi_1(\alpha^{-j}(v), x)$, and

$\psi_2(\alpha^j(v), x) = \psi_1(x, \alpha^{-j}(v))$ for all $x \in \mathcal{R}$ $(j = 0, 1, \cdots, n-1)$.

(4) $\psi_1(x,y) = \psi_1(y,x) = 0 = \psi_2(x,y) = \psi_2(y,x)$ for all $x \in \mathcal{S}, y \in \mathcal{R}$.

By these relations (1),(2),(3) and (4), we obtain some relations between θ_1 and θ_2 as follows. Let $\mathbf{S}_k, \mathbf{M}_k$ and $\mathbf{R}_k$ $(k = 1, 2)$ be the subspaces of $\{\mathcal{S} \oplus (\mathcal{M} + \mathcal{R})\} \otimes \mathbb{C}$ complexifications of $\mathcal{S}, \mathcal{M}$ and $\mathcal{R}$ respectively. We have the involution^{-}: $\mathbf{S} \oplus (\mathbf{M} + \mathbf{R}) \to \mathbf{S} \oplus (\mathbf{M} + \mathbf{R})$ corresponding to the conjugation in the factor $\mathbb{C}$ of the tensor product. The image of $x \in \mathbf{S} \oplus (\mathbf{M} + \mathbf{R})$ under this involution is denoted by $\bar{x}$. Then we have:

(1) $\theta_2(x,y) = \theta_1(x,y)$ for all $x, y \in \mathbf{S} \oplus \mathbf{M}$.

(2) $\theta_2(x,y) = \theta_1(y,x) = \overline{\theta_1(x,y)}$ for all genarators $x, y \in \mathbf{R}$, and

$\theta_2(x,y) = \overline{\theta_1(\bar{x},\bar{y})}$ for all $x, y \in \mathbf{R}$.

(3) $\theta_1(x,y) = \theta_1(\alpha^j(x), \alpha^j(y))$ for all generator $x, y \in \mathbf{M} + \mathbf{R}$,

$\theta_2(x, \alpha^j(v)) = \theta_1(\alpha^{-j}(v), x)$ for every generator x of $\mathbf{R}$,

$\theta_2(\alpha^j(x), v) = \overline{\theta_1(\alpha^j(x), v)}$ for every generator x of $\mathbf{R}$, and

$\theta_2(x, v) = \theta_2(\alpha^j(x), \alpha^{-j}(v))$ for every generator $x \in \mathbf{R}$.

(4) $\theta_k(x,y) = 0$ for all $x \in \mathbf{S}, y \in \mathbf{R}, k = 1, 2$.

For the Hermitian matrices H_{L_k} $(k = 1, 2)$ representing θ_k, we choose a basis of $H_1(F_{L_k}; \mathbb{C})$ from generators of $H_1(F_{L_k}; \mathbb{Z})$. Set $\omega_j = e^{2\pi i \frac{j}{n}}$ $(j = 1, \cdots, n)$. We replace the generating set $\{\alpha^j(v)|j=0,1,\cdots,n-1\}$ of $\mathbf{M}$ by $\{\mathbf{v}_j|\mathbf{v}_j=\sum_{l=0}^{n-1}\omega_j^l\alpha^l(v),\ j=0,1,\cdots,n-1\}$. For $\mathbf{R}$ we consider two choices of generating sets related by involution $^-$. We replace $\{\alpha^j(y_p)|y_p\in X_{\mathcal{R}}, j=0,1,\cdots,n-1\}$ by $\{\mathbf{y}_{j,p}|\mathbf{y}_{j,p}=\sum_{l=0}^{n-1}\omega_j^l\alpha^l(y_p), y_p\in X_{\mathcal{R}}, j=0,1,\cdots,n-1\}$ or $\{\overline{\mathbf{y}_{j,p}}|\overline{\mathbf{y}_{j,p}}=\sum_{l=0}^{n-1}\overline{\omega_j}^l\alpha^l(y_p), y_p\in X_{\mathcal{R}}, j=0,1,\cdots,n-1\}$.

Then we obtain the new generating set for M_k+R_k. This generating set satisfies the following conditions.

(1) $\theta_k(\mathbf{v}_j,\mathbf{v}_m)=0$ for $j\neq m$, where $\mathbf{v}_j,\mathbf{v}_m\in\mathbf{M},\ k=1,2$,

$\theta_1(\mathbf{x}_{j,p},\mathbf{v}_m)=\theta_2(\overline{\mathbf{x}_{j,p}},\mathbf{v}_m)=0$ for $j\neq m$, where $\mathbf{x}_{j,p}\in\mathbf{R}_1,\overline{\mathbf{x}_{j,p}}\in\mathbf{R}_2,\mathbf{v}_m\in\mathbf{M}$,

$\theta_1(\mathbf{x}_{j,p},\mathbf{y}_{m,q})=\theta_2(\overline{\mathbf{x}_{j,p}},\overline{\mathbf{y}_{m,q}})$ for $j\neq m$ where $\mathbf{x}_{j,p},\mathbf{y}_{m,q}\in\mathbf{R}_1,\overline{\mathbf{x}_{j,p}},\overline{\mathbf{y}_{m,q}}\in\mathbf{R}_2$.

(2) $\theta_1(\mathbf{x},\mathbf{y}_{j,p})=\theta_2(\mathbf{x},\overline{\mathbf{y}_{j,p}})=0$ for any $\mathbf{x}\in\mathbf{S},\mathbf{y}_{j,p}\in\mathbf{R}_1,\overline{\mathbf{y}_{j,p}}\in\mathbf{R}_2$.

(3) $\theta_1(\mathbf{y}_{j,p},\mathbf{y}_{j,q})=\overline{\theta_2(\overline{\mathbf{y}_{j,p}},\overline{\mathbf{y}_{j,q}})}$ for any $\mathbf{y}_{j,p},\mathbf{y}_{j,q}\in\mathbf{R}_1,\overline{\mathbf{y}_{j,p}},\overline{\mathbf{y}_{j,q}}\in\mathbf{R}_2$.

(4) $\theta_1(\mathbf{v}_j,\mathbf{y}_{j,p})=\overline{\theta_2(\mathbf{v}_j,\overline{\mathbf{y}_{j,p}})}$ for any $\mathbf{v}_j\in\mathbf{M},\mathbf{y}_{j,p}\in\mathbf{R}_1,\overline{\mathbf{y}_{j,p}}\in\mathbf{R}_2$,

$\theta_1(\mathbf{v}_j,\mathbf{v}_j)=\theta_2(\mathbf{v}_j,\mathbf{v}_j)$ for any $\mathbf{v}_j\in\mathbf{M}$.

(5) $\theta_1(\mathbf{x},\mathbf{v}_j)=\theta_2(\mathbf{x},\mathbf{v}_j)$ for any $\mathbf{x}\in\mathbf{S},\mathbf{v}_j\in\mathbf{M}$.

Take the subspace W_j of $\mathbf{M}\oplus\mathbf{R}$ corresponding to ω_j, and choose its ordered basis by taking $\mathbf{v}_j$ from $\mathbf{M}$ first and $\mathbf{y}_{j,p}$ from $\mathbf{R}$ in arbitrary order. To obtain the ordered basis of $\mathbf{M}\oplus\mathbf{R}$ we place the basis of W_j before the basis of W_{j+1} for $j=0,1,\cdots,n-1$. Finally we add ordered basis of $\mathbf{S}$. Then we have an orderd basis of $H_1(F_L;\mathbb{C})$. We also have an ordered basis of $H_1(F_{L_2};\mathbb{C})$ by replacing each $\mathbf{y}_{j,p}$ with $\overline{\mathbf{y}_{j,p}}$.

We obtain the matrices of forms θ_1 and θ_2 in ordered bases of $\mathbf{S}\oplus\mathbf{M}\oplus\mathbf{R}$ as described below.

$$H'_{L_1} = \begin{pmatrix} B_{10} & & \mathbf{0} & {}^t\overline{S_0} \\ & \ddots & & \vdots \\ \mathbf{0} & & B_{1n-1} & {}^t\overline{S_{n-1}} \\ S_0 & \cdots & S_{n-1} & S \end{pmatrix}, H'_{L_2} = \begin{pmatrix} B_{20} & & \mathbf{0} & {}^t\overline{S_0} \\ & \ddots & & \vdots \\ \mathbf{0} & & B_{2n-1} & {}^t\overline{S_{n-1}} \\ S_0 & \cdots & S_{n-1} & S \end{pmatrix}.$$

In those bases, B_{1j} (B_{2j} respectively), where $j = 0, 1, \cdots, n-1$, is the matrix of the restriction of the form θ_1 (θ_2 respectively) to the subspace W_j generated by $\{\mathbf{v}_j\} \cup \{\mathbf{y}_{j,p} \mid y_p \in X_{\mathcal{R}_1}\}$ ($\{\mathbf{v}_j\} \cup \{\overline{\mathbf{y}_{j,p}} \mid y_p \in X_{\mathcal{R}_1}\}$ respectively). Finally S is the restriction to the stator part which is the same for θ_1 and θ_2. Notice that ${B_{1k}}^t = B_{2k}$, $S_l = (\mathbf{s}_{l1}\ \mathbf{0} \cdots \mathbf{0})$, and $\mathbf{s}_{l1}$ is the first column of each matrix S_l.

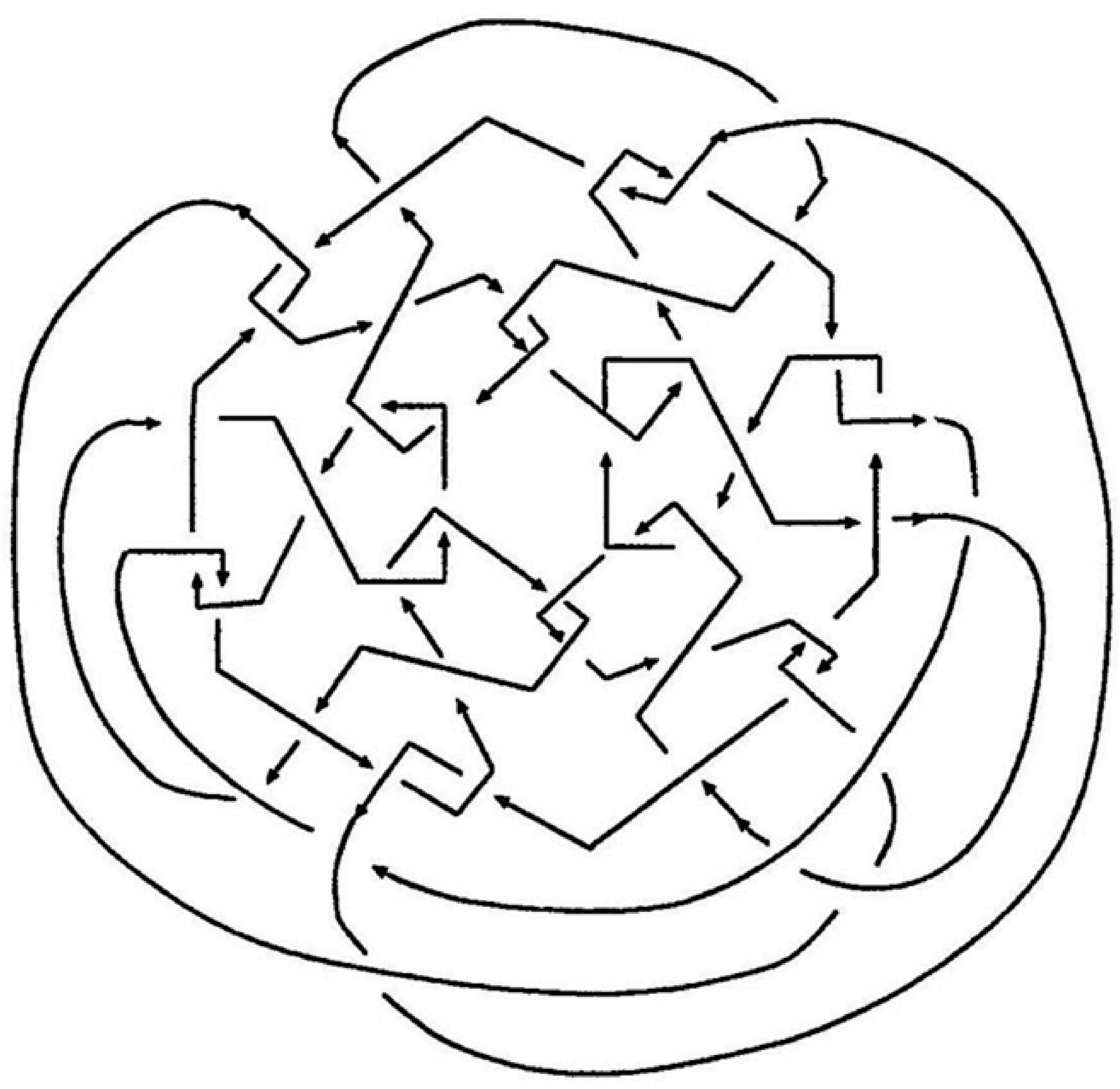
Matrices $M_k = (H'_{L_k} - \lambda E)$ $(k = 1, 2)$ satisfy the conditions of Traczyk's Proposition 2.9 in [38] for any real number λ. Thus $\det(M_1) = \det(M_2)$ for any real λ. So $\det(M_1) = \det(M_2)$ for any complex λ as well.

□

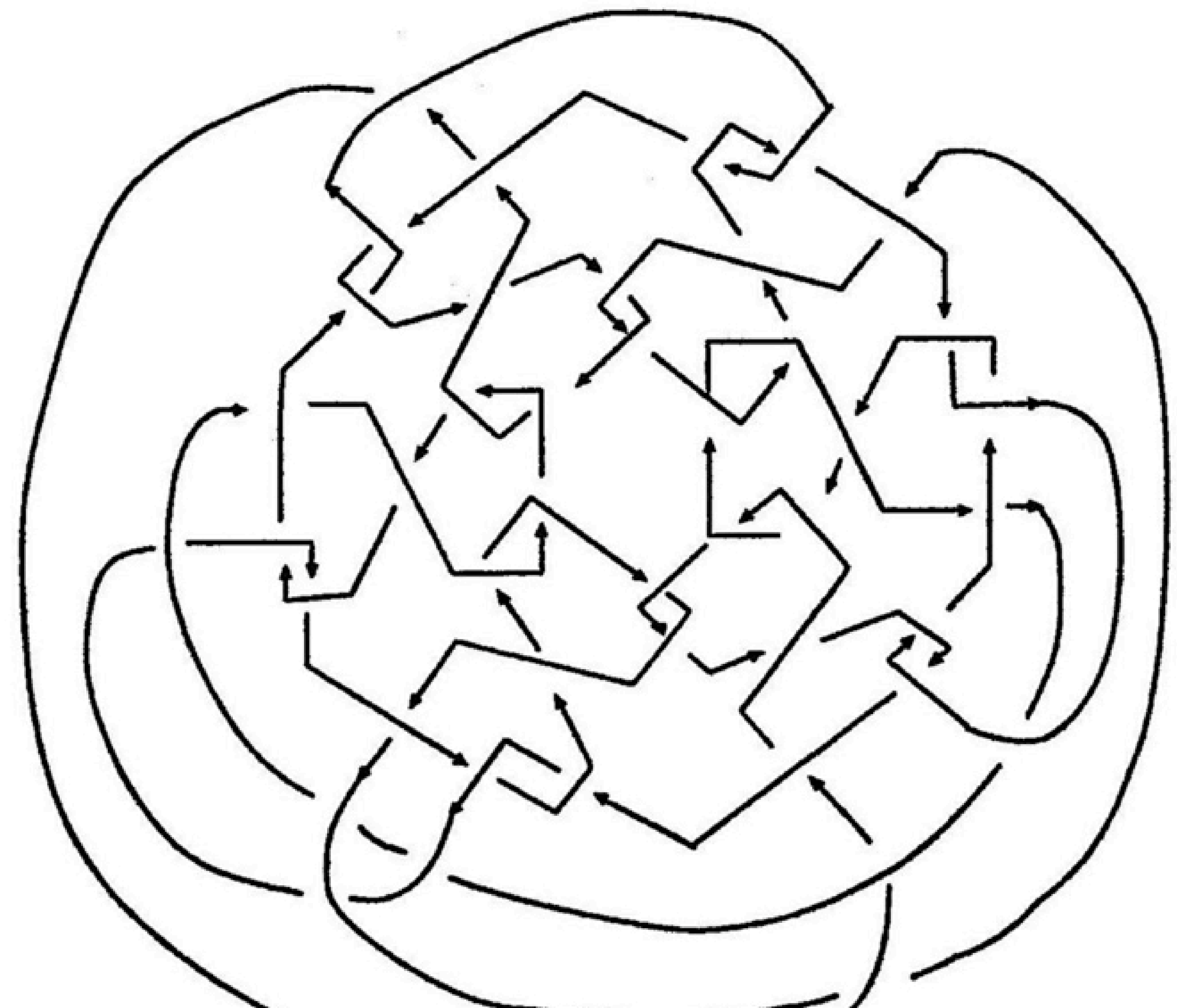
3.5 Rotation and the Alexander polynomial

In [3], it is shown that any pair of oriented three or four-rotant links share the same Alexander polynomial (in particular, Homflypt polynomial). Traczyk [38] showed that any pair of orientation-preserving n-rotant links share the same Alexander polynomial. On the other hand, it is not known whether orientation-reversing n-rotants ($n \geq 6$) preserve the Alexander polynomial. Now, we present an example of a pair of 6-rotant links which share neither the same Alexander polynomial nor the same Jones polynomial. Therefore, the invariance in [3] of Alexander polynomial and the Jones polynomial for the orientation-reversing rotant links are the best possible.

Let L_1 and L_2 be the knots illustrated in Fig.3.12.



L_1



L_2

Fig.3.12

Then, by using KNOT [24], we have the following.

Alexander polynomial :

$\Delta_{L_1}(t) = -t^{12} + 20t^{11} - 198t^{10} + 1252t^{9} - 5557t^{8} + 18146t^{7} - 44725t^{6} + 84224t^{5} - 121232t^{4} + 131689t^{3} - 105284t^{2} + 62756t - 42179 + 62756t^{-1} - 105284t^{-2} + 131689t^{-3} - 121232t^{-4} + 84224t^{-5} - 44725t^{-6} + 18146t^{-7} - 5557t^{-8} + 1252t^{-9} - 198t^{-10} + 20t^{-11} - t^{-12}$,

and

$\Delta_{L_2}(t) = -t^{12} + 16t^{11} - 137t^{10} + 814t^{9} - 3619t^{8} + 12305t^{7} - 32213t^{6} + 65112t^{5} - 101765t^{4} + 123334t^{3} - 117939t^{2} + 96506t - 84825 + 96506t^{-1} - 117939t^{-2} + 123334t^{-3} - 101765t^{-4} + 65112t^{-5} - 32213t^{-6} + 12305t^{-7} - 3619t^{-8} + 814t^{-9} - 137t^{-10} + 16t^{-11} - t^{-12}$.

Jones polynomial :

$V_{L_1}(t) = t^{23} - 16t^{22} + 131t^{21} - 713t^{20} + 2881t^{19} - 9193t^{18} + 24058t^{17} - 52926t^{16} + 99534t^{15} - 161854t^{14} + 229195t^{13} - 283357t^{12} + 304679t^{11} - 280476t^{10} + 211413t^{9} - 112418t^{8} + 7697t^{7} + 77824t^{6} - 127092t^{5} + 136195t^{4} - 114114t^{3} + 77214t^{2} - 41391t + 16087 - 2934t^{-1} - 1501t^{-2} + 1760t^{-3} - 954t^{-4} + 343t^{-5} - 84t^{-6} + 13t^{-7} - t^{-8}$,

and

$V_{L_2}(t) = t^{23} - 16t^{22} + 131t^{21} - 713t^{20} + 2881t^{19} - 9193t^{18} + 24057t^{17} - 52919t^{16} + 99503t^{15} - 161752t^{14} + 228932t^{13} - 282808t^{12} + 303730t^{11} - 279098t^{10} + 209727t^{9} - 110701t^{8} + 6314t^{7} + 78540t^{6} - 126958t^{5} + 135242t^{4} - 112578t^{3} + 75451t^{2} - 39756t + 14823 - 2118t^{-1} - 1933t^{-2} + 1941t^{-3} - 1010t^{-4} + 354t^{-5} - 85t^{-6} + 13t^{-7} - t^{-8}$.

4 Branched covers of tangles in three-balls

A p-fold branched cover of an n-tangle is a three-manifold, the boundary of which is a connected surface of genus $(n-1)(p-1)$. Such a manifold can be obtained from the genus $(n-1)(p-1)$ handlebody by a surgery. First, we provide an algorithm for a surgery description of a p-fold cyclic branched cover of B^3 branched along a tangle in section 4.1. The construction generalizes that of Montesinos [27] and Akbulut and Kirby [2]. It is strikingly simple in the case of a two-fold branched cover. In section 4.2, we also discuss the related Heegaard decomposition of a p-fold branched cover of an n-tangle.

4.1 Surgery descriptions

Let T be an n-tangle and T_0 a trivial n-tangle diagram (Fig.4.1).

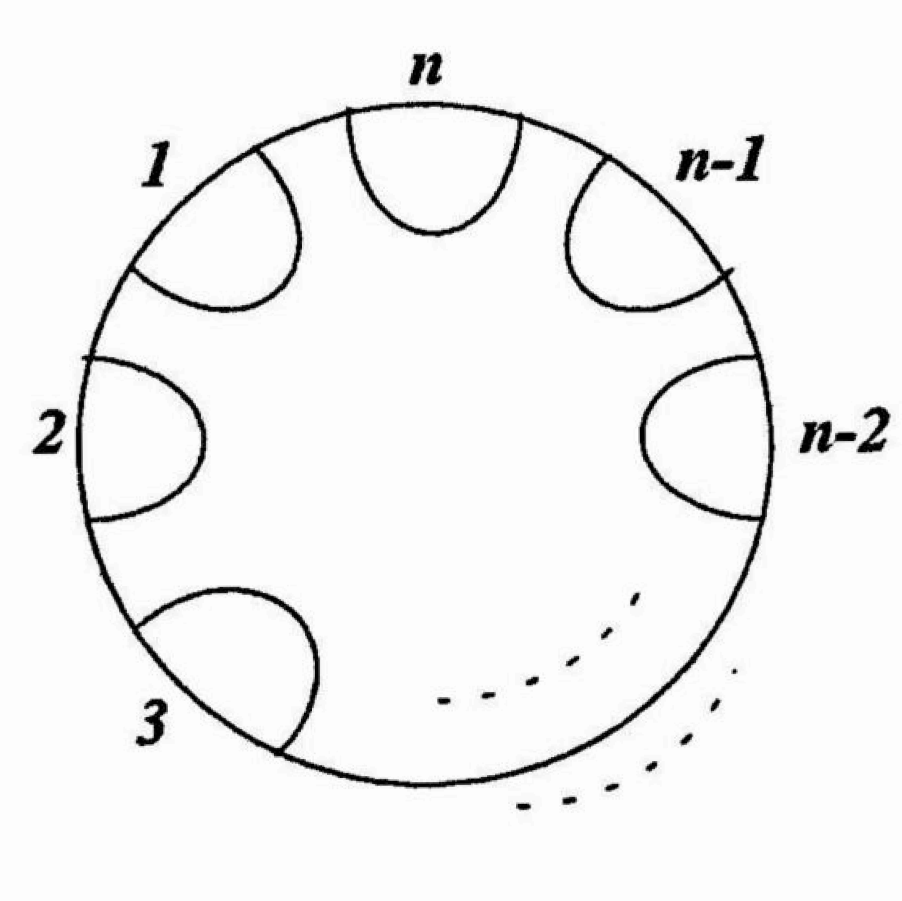


Fig. 4.1

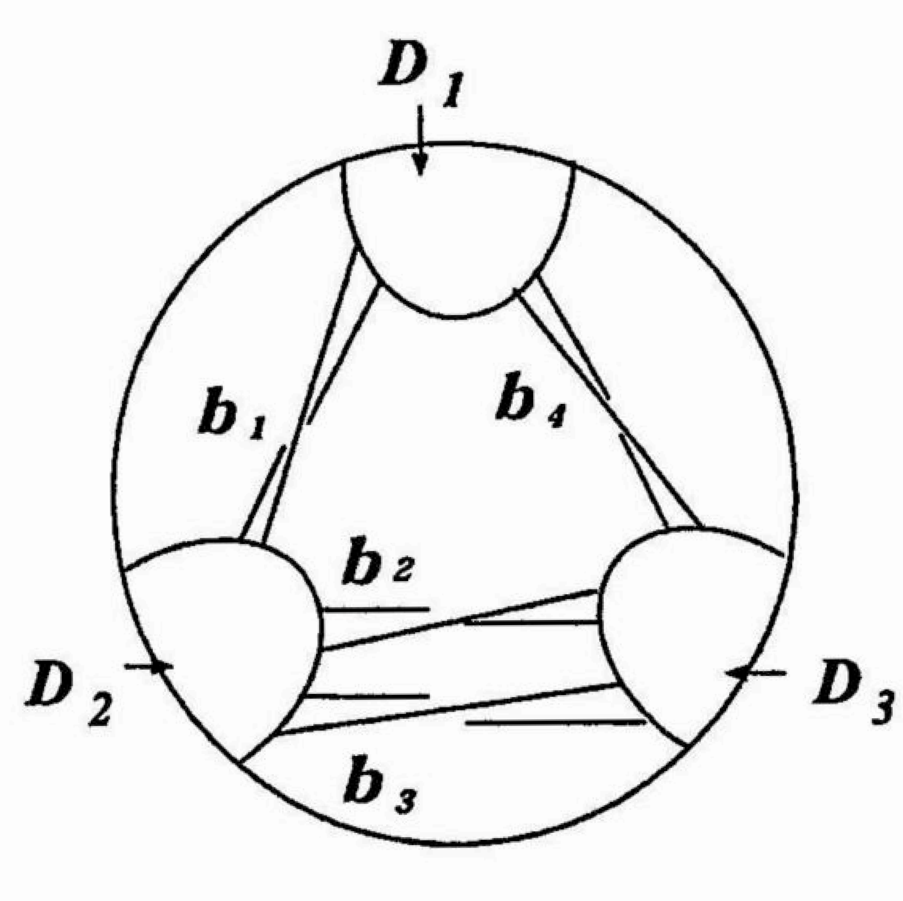


Fig. 4.2

Let $D_1 \cup \cdots \cup D_n$ be a disjoint union of disks bounded by T_0 and $b_1, \cdots, b_m$ be mutually disjoint disks in B^3 such that $b_i \cap \bigcup_j D_j = \partial b_i \cap T_0$ are two disjoint arcs in $\partial b_i (i = 1, \cdots, m)$ (see Fig.4.2). We denote by $\Omega(T_0; \{D_1, \cdots, D_n\}, \{b_1, \cdots, b_m\})$

the tangle $T_0 \cup \bigcup_i \partial b_i - \mathrm{int}(T_0 \cap \bigcup_i \partial b_i)$ and call it a *disk-band representation* of a tangle. A disk-band representation is called *bicollared* if the surface $\bigcup_i D_i \cup \bigcup_j b_j$ is orientable. We will see that any n-tangle has a bicollared disk-band representation in Proposition 4.1.

A *framed link* is a disjoint union of embedded annuli in a three-manifold. Framed links in S^3 can be identified with links whose each component is assigned an integer. Such links are also called framed links. Let M be a three-manifold and $\mathcal{L}$ a framed link in M. We denote by $\Sigma(\mathcal{L}, M)$ the manifold obtained from M by the surgery along $\mathcal{L}$ (see [31]).

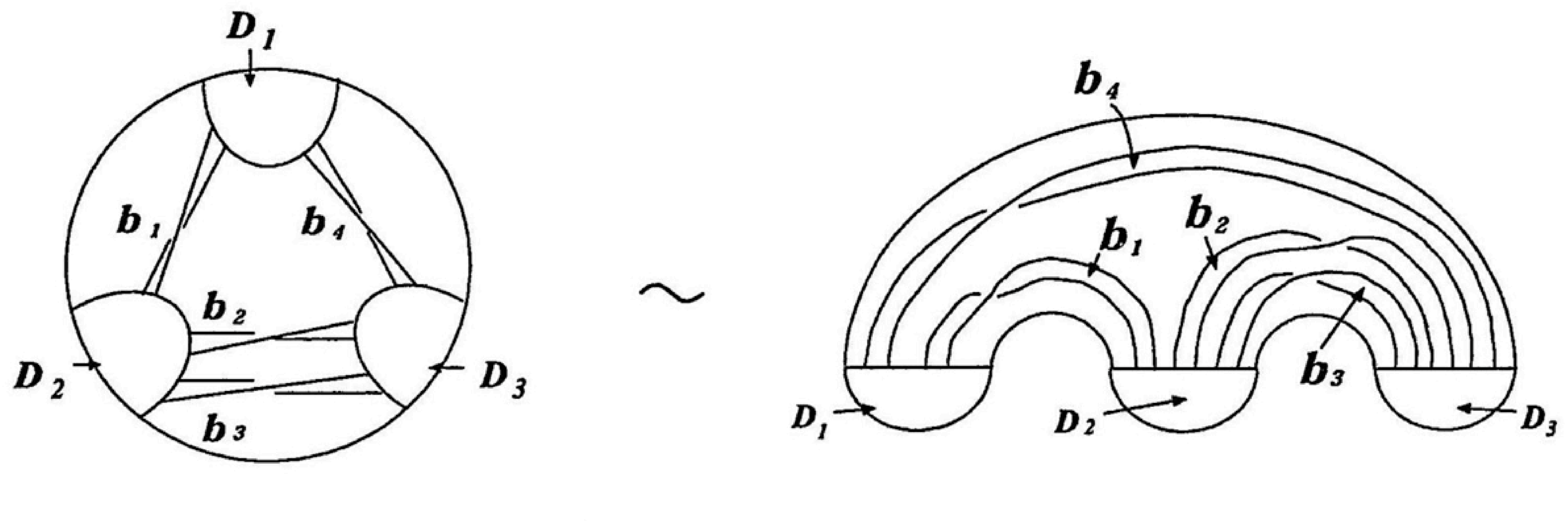
The case of two-fold branched covers is easy to visualize so we will formulate it first.

Theorem 4.1 *Let $\Omega(T_0; \{D_1, \cdots, D_n\}, \{b_1, \cdots, b_m\})$ be a disk-band representation of an n-tangle T in B^3. Let $\varphi : H_0 \to B^3$ be the two-fold branched cover of B^3 by a genus $n-1$ handlebody H_0 branched along T_0. Then the two-fold branched cover of B^3 branched along T has a surgery description $\Sigma(\varphi^{-1}(\bigcup_i b_i), H_0)$ (see* Fig.4.3).

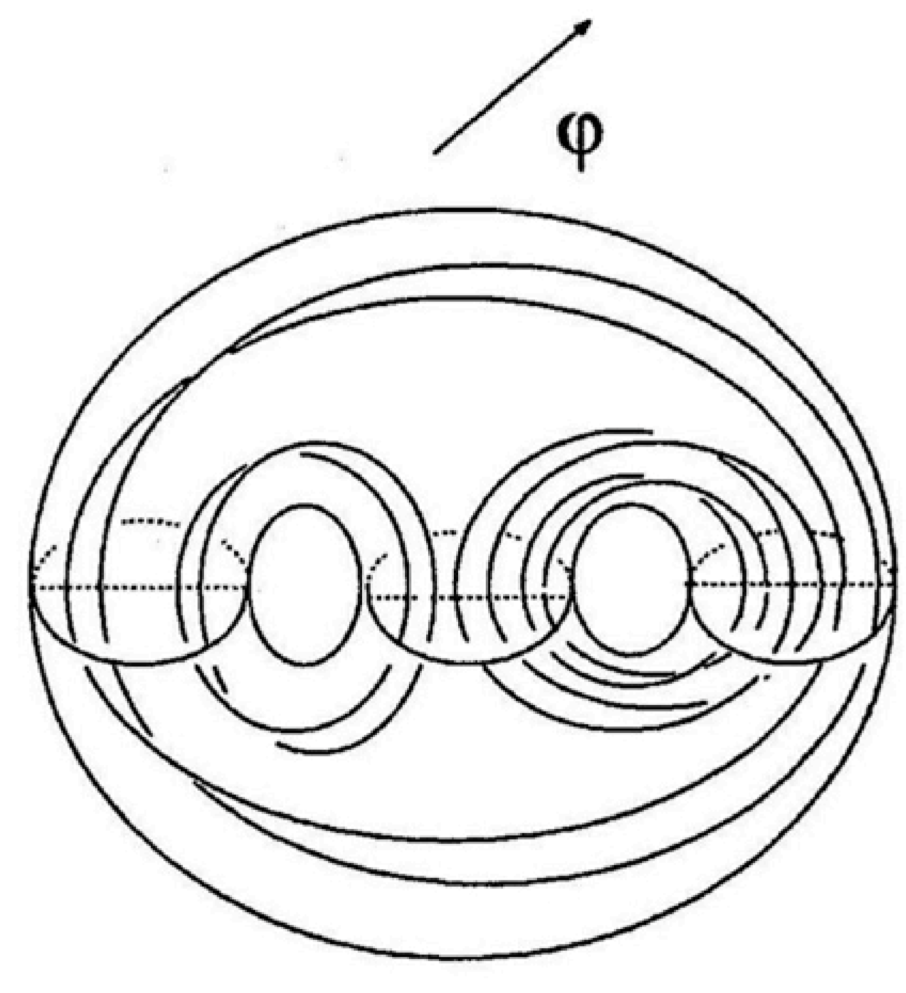
Proof Let X be $B^3 - \bigcup_i D_i$ compactified with two copies, $D_i^{\pm}$ of D_i $(i = 1, 2, \cdots, n)$ as in Fig. 4.4. Let X_1 and X_2 be two copies of X, and let $D_{i,k}^{\pm} \subset X_k$ denote copies of $D_i^{\pm}$ $(i = 1, 2, \cdots, n,\ k = 1, 2)$ as in Fig. 4.5. Then H_0 is obtained from $X_1 \cup X_2$ by identifying $D_{i,1}^{\varepsilon}$ with $D_{i,2}^{-\varepsilon}$ $(\varepsilon \in \{-, +\})$. Let $b_{j,k} = \varphi^{-1}(b_j) \cap X_k$ and let Y be $H_0 - \bigcup_{j,k} b_{j,k}$ compactified with two copies $b_{j,k}^{\pm}$ of $b_{j,k}$ in X_k $(j = 1, 2, \cdots, m,\ k = 1, 2)$. Here, $+$ or $-$ sides of $D_{i,k}$ and $b_{j,k}$ are not necessarily compatible. We note, and it is the key observation of the construction, that the two-fold branched cover H of B^3 branched along T is obtained from Y by identifying $b_{j,1}^{\varepsilon}$ with $b_{j,2}^{-\varepsilon}$ $(\varepsilon \in \{-, +\})$. Note that each $b_{j,1}^{+} \cup b_{j,2}^{-} \cup b_{j,1}^{-} \cup b_{j,2}^{+}$ is a torus. Let c_j be the core of the annulus $b_{j,1}^{+} \cup b_{j,2}^{-}$. The manifold obtained from Y by identifying $b_{j,1}^{\varepsilon}$ with $b_{j,2}^{-\varepsilon}$ is homeomorphic to the one obtained from Y by attaching tori $D_j^2 \times S^1$ $(j = 1, 2, ..., m)$ so that $\partial D_j^2 = c_j$. Hence

H is homeomorphic to the manifold with the surgery description $\Sigma(\varphi^{-1}(\bigcup_j b_j), H_0)$.

$\square$



$\Omega(T_0; \{D_1, D_2, D_3\}, \{b_1, b_2, b_3, b_4\})$



$\Sigma(\varphi^{-1}(\bigcup_i b_i), H_0)$

Fig. 4.3

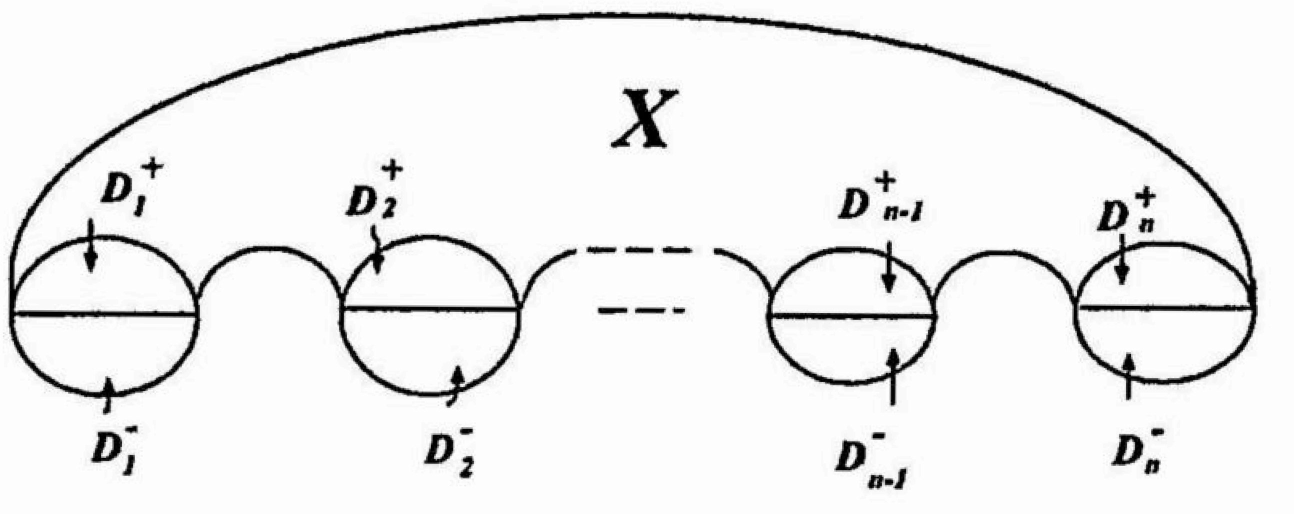


Fig. 4.4

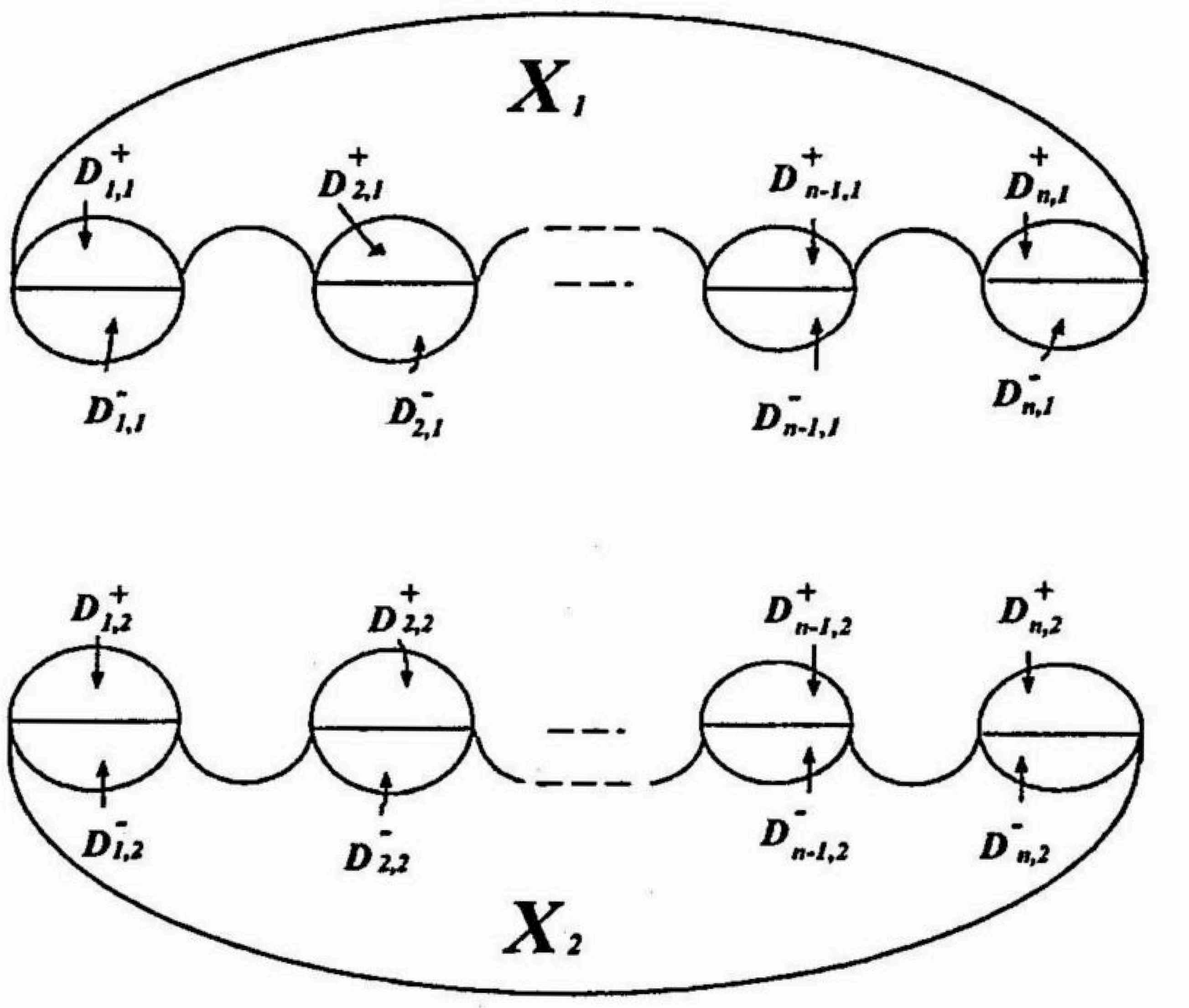


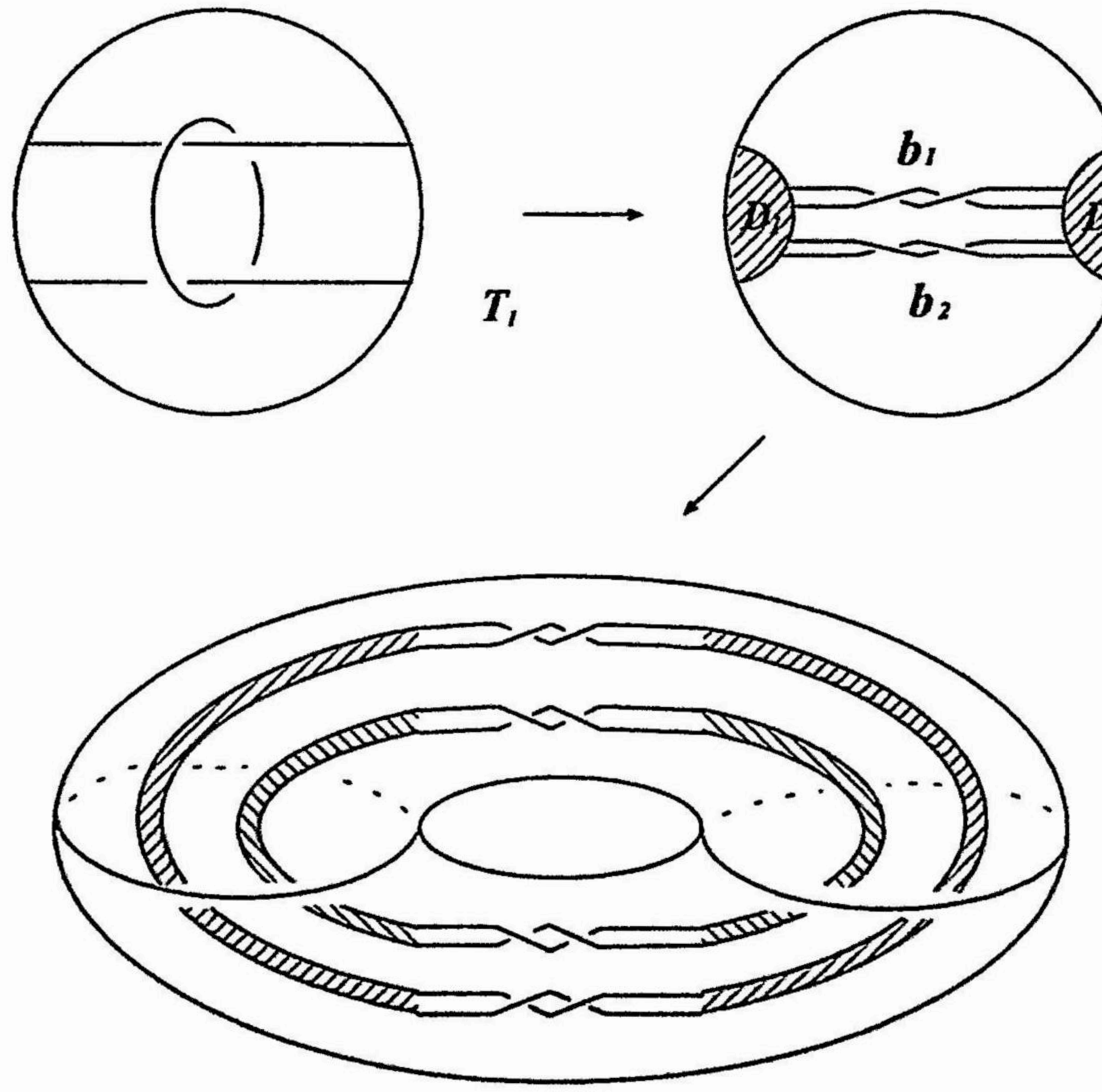
Fig. 4.5

Example 4.1 (a) The two-fold branched cover $M^{(2)}(T_1)$ branched along a tangle T_1 in Fig. 4.6 is the Seifert manifold with the base a disk and two special fibers of type $(2,1)$ and $(2,-1)$. Furthermore $M^{(2)}(T_1)$ is a twisted I-bundle over the Klein bottle (for example see [16]). In particular, $\pi_1(M^{(2)}(T_1)) = \langle a, b \mid aba^{-1}b = 1 \rangle$.

(b) If we glue together two copies of T_1 as in Fig. 4.7, we get Borromean rings L. Thus our previous computation shows that the two-fold branched cover $M^{(2)}(L)$ of S^3 branched along L is a "switched" double of the twisted I-bundle over the Klein bottle (see Fig. 4.8 for a surgery description). The fundamental group $\pi_1(M^{(2)}(L)) = \langle x, a \mid x^2ax^2a^{-1}, a^2xa^2x^{-1} \rangle$ is a three-manifold group which is torsion free but not left orderable (see [36]).

(c) If we take the double of the tangle T_1, we obtain the link in Fig. 4.9. The two-fold branched cover of S^3 branched along this link is the double of twisted I-bundle over Klein bottle. A surgery description of this manifold is shown in Fig. 4.10. Thus this manifold is the Seifert manifold with the base S^2 and four special fibers of type $(2,1),(2,1),(2,-1)$ and $(2,-1)$. This manifold also has another Seifert fibration, which is a circle bundle over the Klein bottle.

Example 4.1(a) was motivated by the fact that the tangle T_1 yields a virtual Lagrangian of index 2 in the symplectic space of the Fox $\mathbf{Z}$-colorings of the boundary of our tangle in [9].



$M^{(2)}(T_1)$

Fig. 4.6

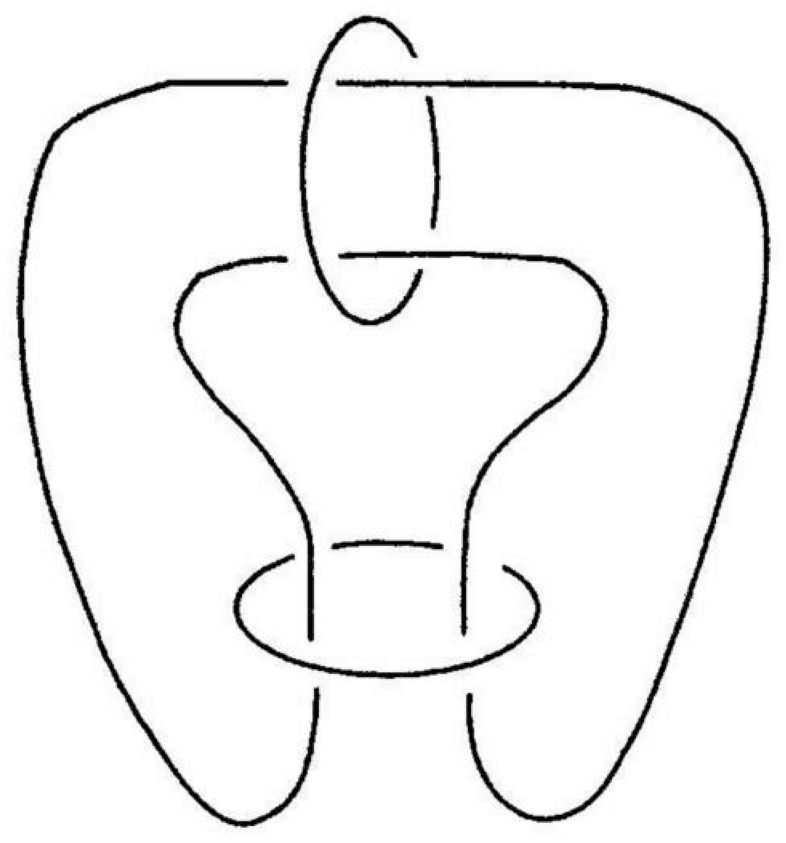


Fig. 4.7

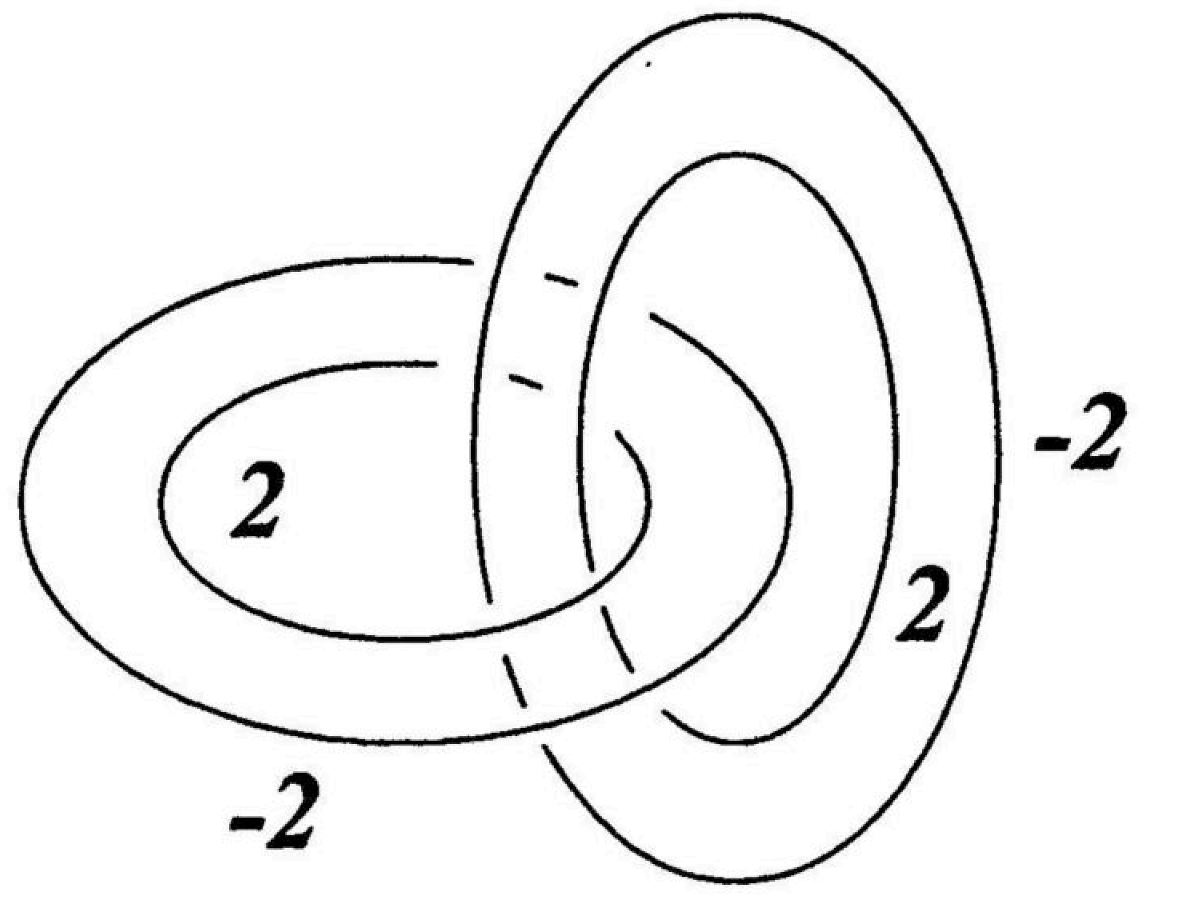


Fig. 4.8

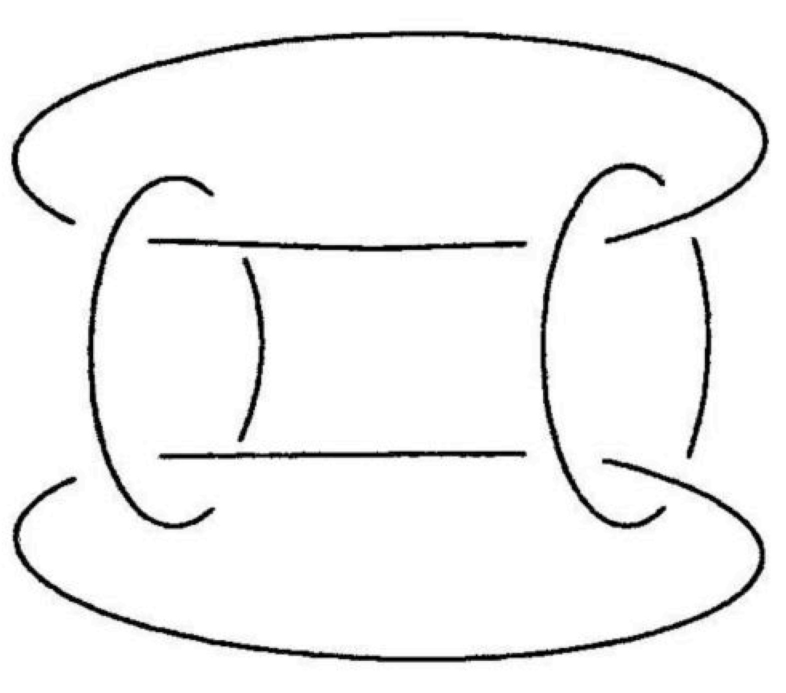


Fig. 4.9

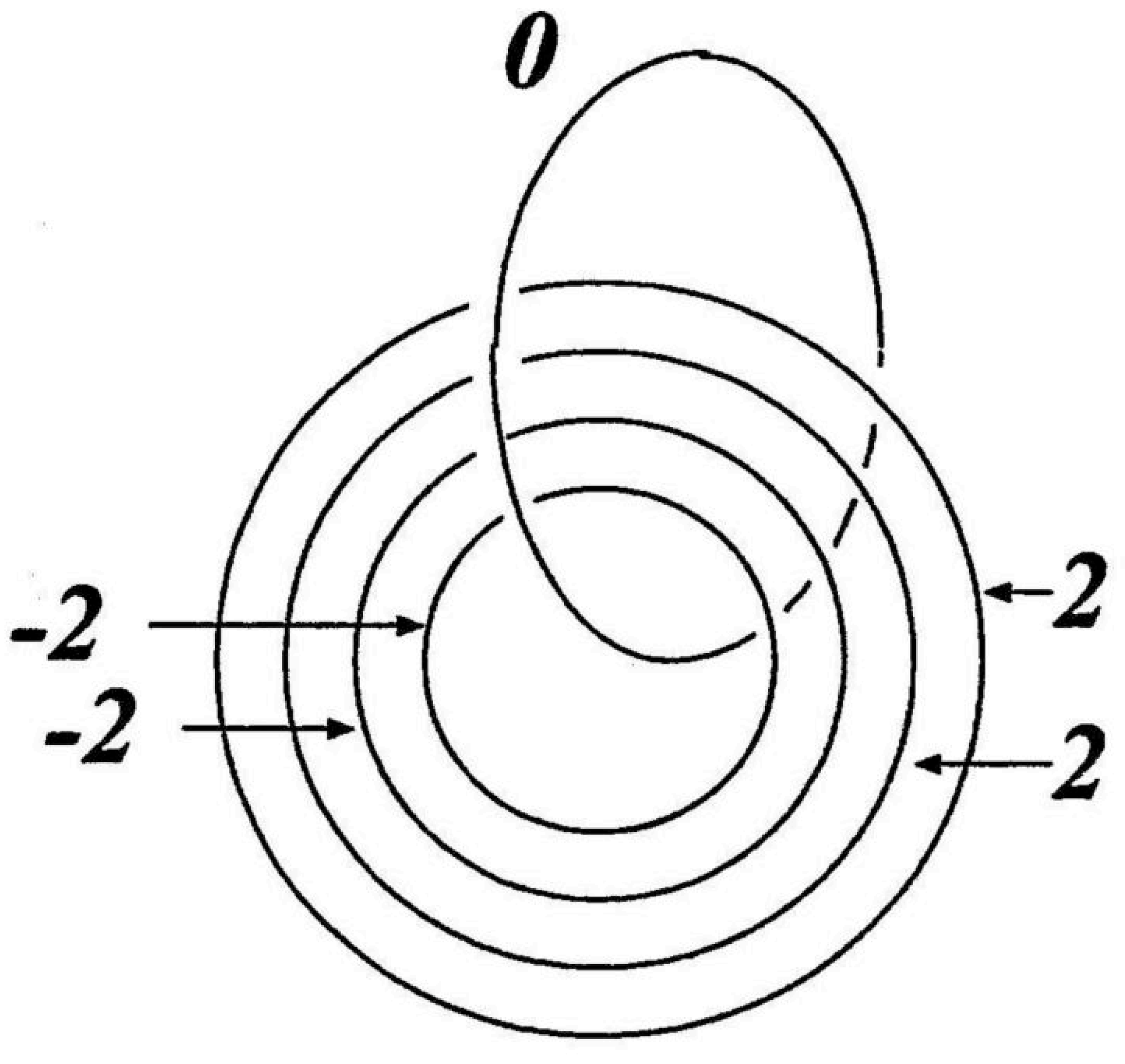


Fig. 4.10

More generally we have:

Example 4.2 Consider a tangle T_2 in Fig.4.11, called a *pretzel tangle* of type $(a_1, a_2, \cdots, a_n)$, where each a_i is an integer indicating the number of half-twists $(i = 1, 2, \cdots, n)$. The two-fold branched cover $M^{(2)}(T_2)$ branched along the tangle T_2 is a Seifert fibered manifold with the base a disk and n special fibers of type $(a_1, 1)$,$(a_2, 1)$, $\cdots$, $(a_n, 1)$ (see Fig.4.12).

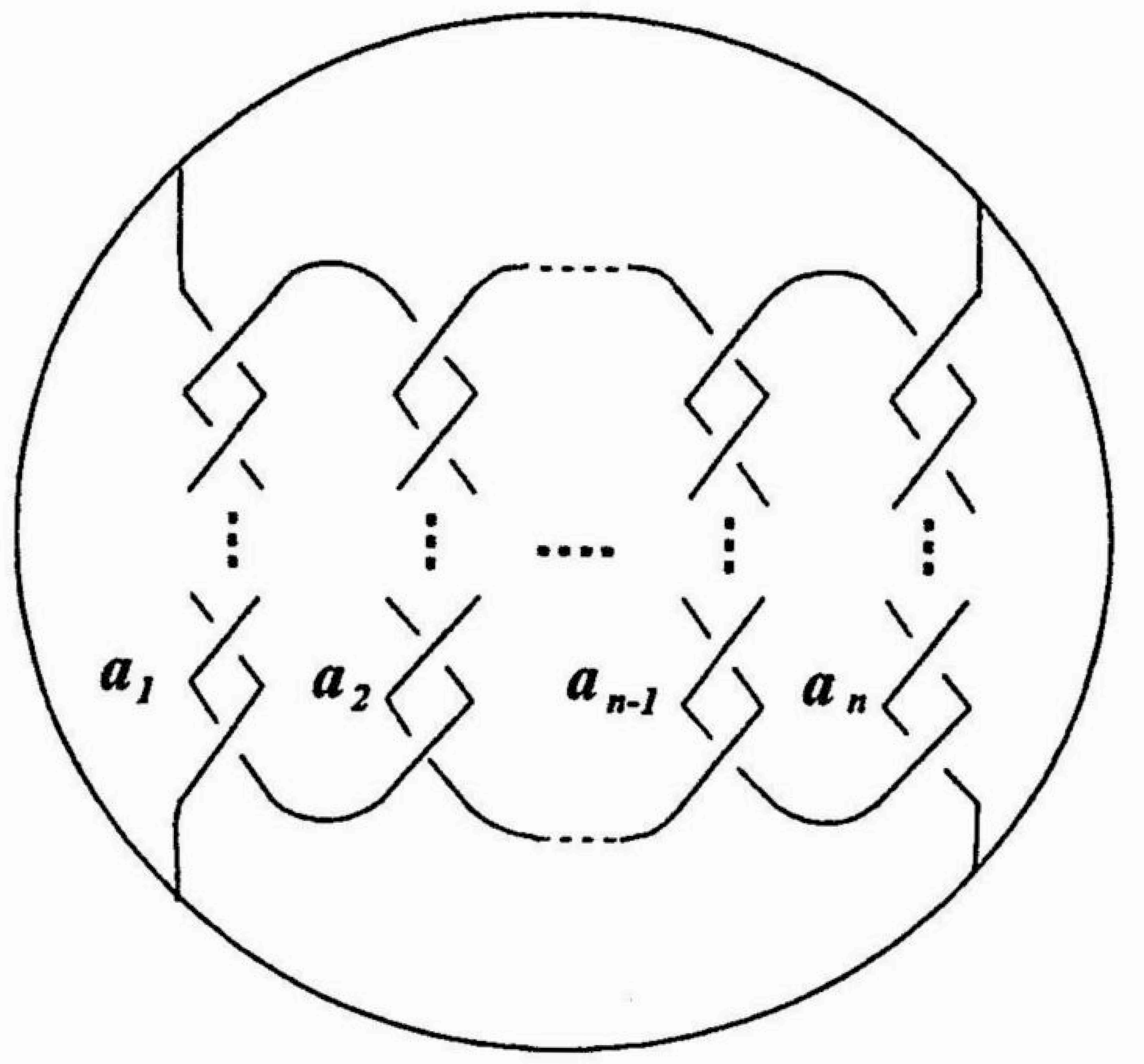


Fig. 4.11

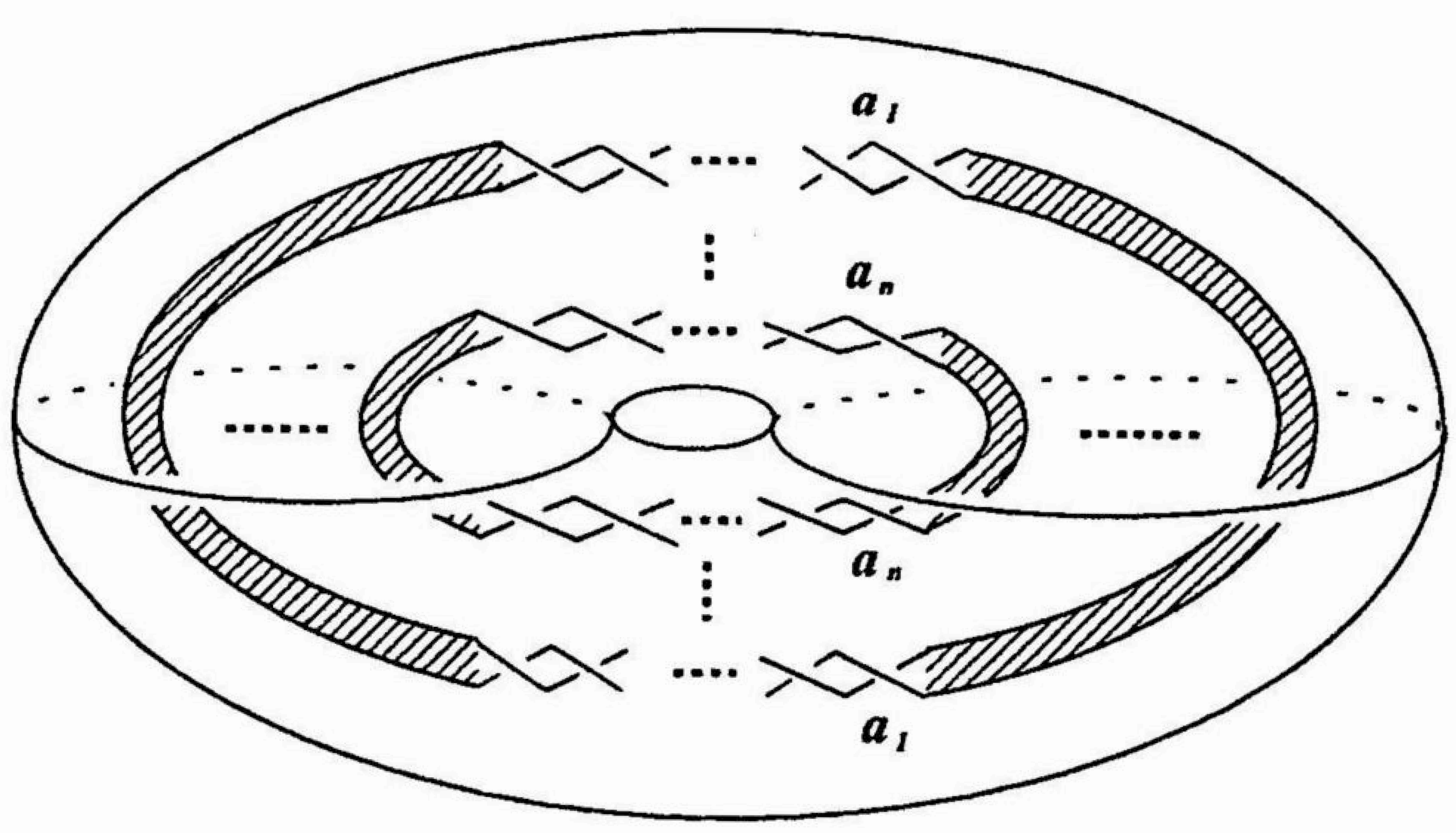


Fig. 4.12

Theorem 4.1 can be generalized to a p-fold cyclic branched cover assuming that an n-tangle is oriented whose disk-band representation is bicollared, where p is any positive integer greater than 2. We proceed as follows:

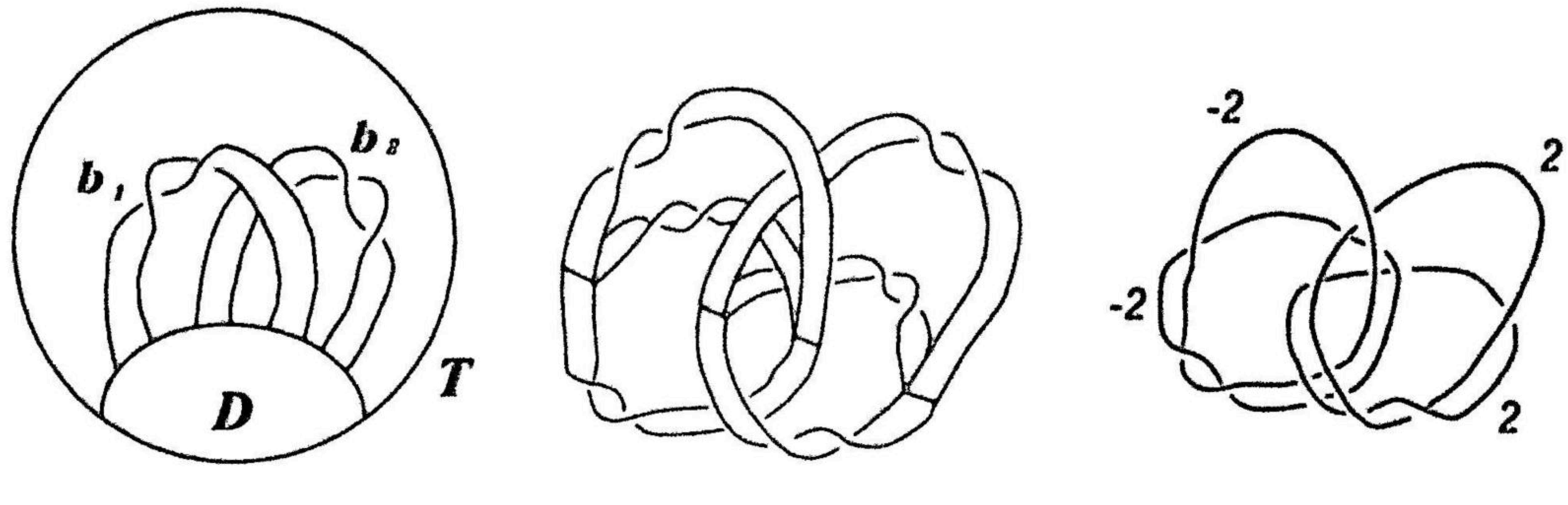
Let $T = \Omega(T_0; \{D_1, \cdots, D_n\}, \{b_1, \cdots, b_m\})$ be a bicollared disk-band representation of an n-tangle. Then $\bigcup_i D_i \cup \bigcup_j b_j$ has a bicollar neighborhood $(\bigcup_i D_i \cup \bigcup_j b_j) \times [-1, 1]$. Let $X = \overline{B^3 - ((\bigcup_i D_i) \times [-1, 1])}$ and $D_i^{\pm} = (D_i \times [\pm 1, 0]) \cap \partial X$. Let X_k be a copy of X and $D_{i,k}^{\pm} \subset \partial X_k$ a copy of $D_i^{\pm} (k = 1, 2, \cdots, p)$. Then the p-fold cyclic branched cover $\varphi : H_0 \to B^3$ branched along T_0 is obtained from $X_1 \cup \cdots \cup X_p$ by identifying $D^+_{i,k}$ with $D^-_{i,k+1}(k = 1, \cdots, p)$, where k is considered modulus p. Let $b^{\pm}_{j,k} = \varphi^{-1}(b_j \times \{\pm 1\}) \cap X_k$. Note that each $b^+_{j,k} \cup b^-_{j,k+1}$ is an annulus in H_0 for any j and k. Then we obtain the p-fold cyclic branched cover of B^3 branched along T in a similar way as in Theorem 4.1.

Theorem 4.2 *Let $\Omega(T_0; \{D_1, \cdots, D_n\}, \{b_1, \cdots, b_m\})$ be a bicollared disk-band representation of an n-tangle T in B^3. Then $\Sigma(\bigcup_{j=1}^m(\bigcup_{k=1}^{p-1}(b^+_{j,k} \cup b^-_{j,k+1})), H_0)$ is the p-fold cyclic branched cover of B^3 branched along T.*

Note that we do not use the annuli $b^+_{j,p+1} \cup b^-_{j,1}$ $(j = 1, \cdots, m)$ in the above theorem. In fact the cores of these annuli bound mutually disjoint 2-disks in $\Sigma(\bigcup_{j=1}^m(\bigcup_{k=1}^{p-1}(b^+_{j,k} \cup b^-_{j,k+1})), H_0)$.

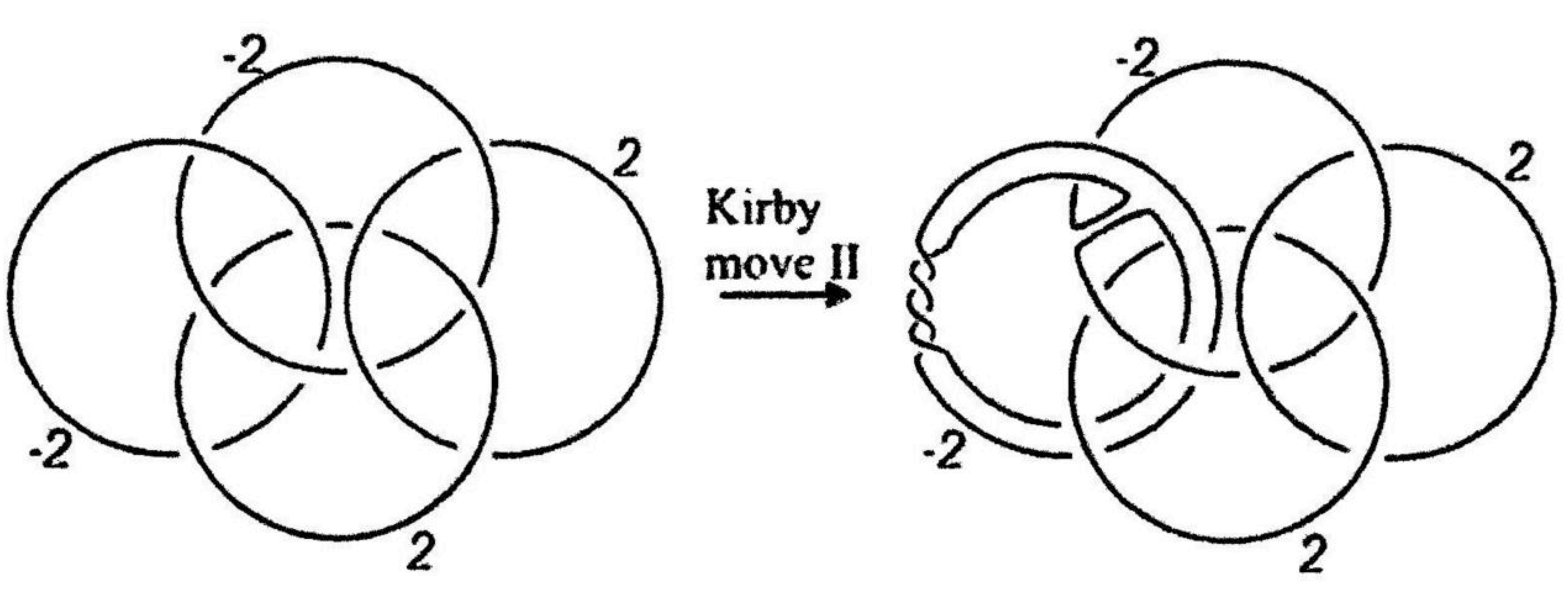
Proof Let $Y = \overline{H_0 - \bigcup_j \varphi^{-1}(b_j \times [-1, 1])}$, $V^{\pm}_{j,k} = \varphi^{-1}(b_j \times [\pm 1, 0]) \cap X_k$ and $\beta^{\pm}_{j,k} = V^{\pm}_{j,k} \cap Y (= \partial V^{\pm}_{j,k} \cap \partial Y)$. Note that $\varphi^{-1}(b_j \times [-1, 1])$ is a genus $p-1$ handlebody. Then the p-fold cyclic branched cover of B^3 branched along T is homeomorphic to a manifold H that is obtained from Y by identifying $\beta^+_{j,k}$ with $\beta^-_{j,k+1}(k = 1, \cdots, p)$, where k is taken modulus p. Moreover H is homeomorphic to a manifold obtained from $\overline{H_0 - \bigcup_j \varphi^{-1}(b_j \times [-1, 1])} \cup \bigcup_j (V^-_{j,1} \cup V^+_{j,p} \cup \varphi^{-1}(b_j \times \{0\}))$ by identifying $\beta^+_{j,k}$ and $b_{j,k}$ with $\beta^-_{j,k+1}$ and $b_{j,k+1} (j = 1, \cdots, m,\ k = 1, \cdots, p-1)$ respectively, where $b_{j,k} = \varphi^{-1}(b_j \times \{0\}) \cap X_k$. Note that $\beta^+_{j,k} \cup b_{j,k} \cup \beta^-_{j,k+1} \cup b_{j,k+1}$ is a torus. By an argument similar to that in the proof of Theorem 4.1, we have the required result. □

Example 4.3 Let $T = \Omega(T_0; \{D\}, \{b_1, b_2\})$ be a tangle as in Fig.4.13(a) and $\varphi : H_0 \to B^3$ the three-fold cyclic branched cover of B^3 branched along T_0. Note that H_0 is a three-ball and $\varphi^{-1}(b_1 \cup b_2)$ is as shown in Fig.4.13(b). By Theorem 4.2, the three-fold cyclic branched cover of B^3 branched along T is obtained from H_0 by the surgery along a framed link in Fig.4.13(c). Note that Fig.4.13(c) is ambient isotopic to Fig.4.14(a). Since the figure eight knot has a tangle decomposition into T and a trivial 1-tangle, the three-fold cyclic branched cover $M^{(3)}(4_1)$ of S^3 branched along the figure eight knot has a surgery description shown in Fig.4.14(a). The framed link in Fig.4.14(a) can be deformed into the link in Fig.4.14(b) by an ambient isotopy and a second Kirby move. The link in Fig.4.14(b) is ambient isotopic to the link in Fig.4.8 (*Fig*.4.14(c)). Hence $M^{(3)}(4_1)$ is homeomorphic to the two-fold branched cover of S^3 branched along the Borromean rings [27, 14](cf. Example 4.1(b)).



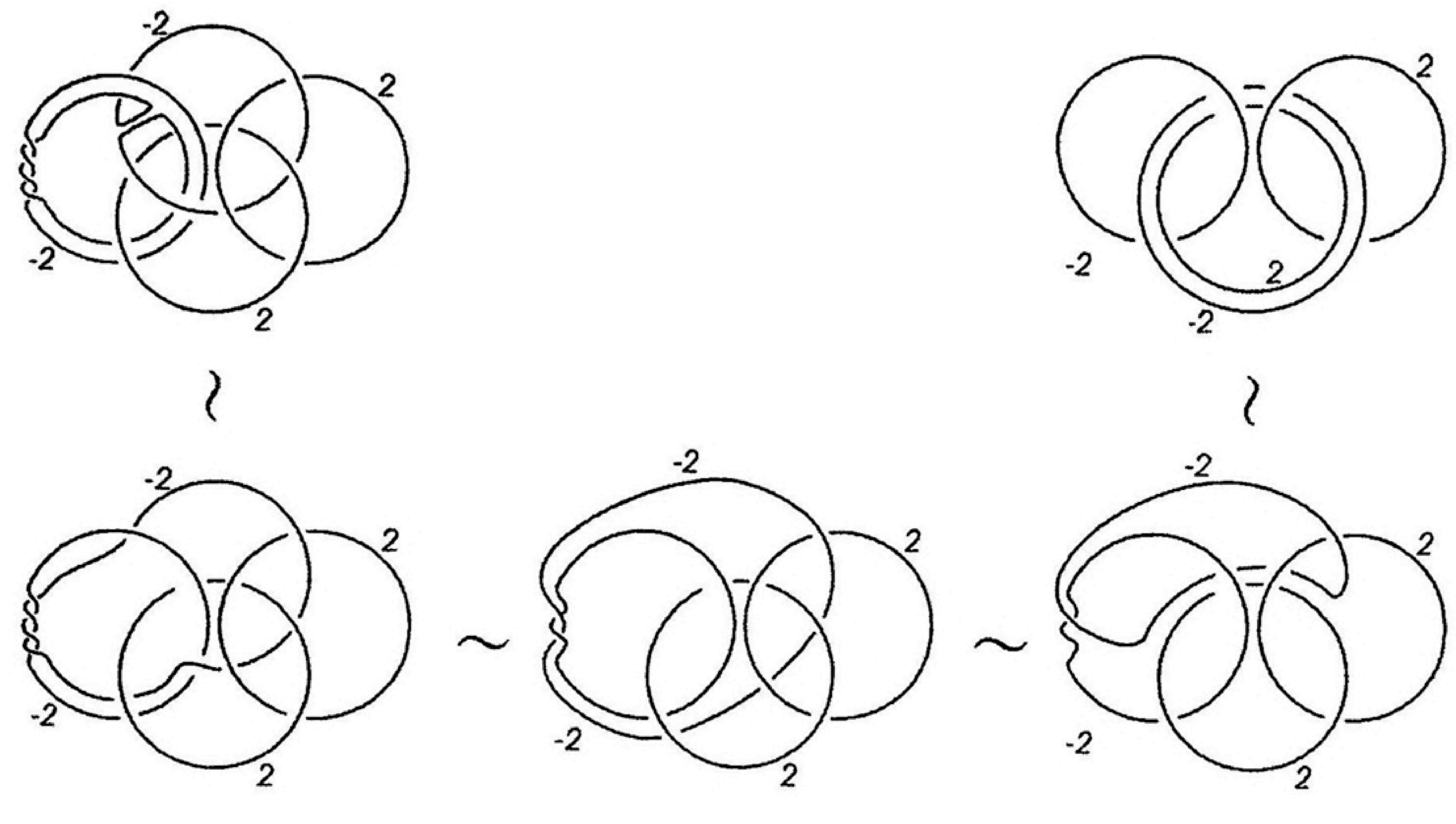
(a) (b) (c)

Fig. 4.13



(a) (b)

Fig.4.14



(c)

Fig. 4.14

Proposition 4.1 *Any n-tangle (B^3, T) has a (bicollared) disk-band representation.*

Proof We attach a trivial n-tangle T_0 to the n-tangle T by a homeomorphism $\varphi : \partial(B, T_0) \to \partial(B, T)$. We obtain a link $L = T_0 \cup T$ in a three-sphere $B \cup_\varphi B$ with a diagram $D(T_0 \cup T)$ as in Fig. 4.15. We may assume that the diagram $D(T_0 \cup T)$ is connected. We color, in checkerboard fashion, the regions of the plane cut by the diagram $D(T_0 \cup T)$ and choose n points $\{v_1, v_2, \cdots, v_n\}$ as in Fig. 4.16. Since $D(T_0 \cup T)$ is connected, there is a spine G of the black surface with the vertex set $V(G)$ containing $\{v_1, v_2, \cdots, v_n\}$. Deforming G on the surface by edge contractions, we have a new spine H with $V(H) = \{v_1, v_2, \cdots, v_n\}$. By retracting the black regions into the neighborhood of H and restricting to B^3, we have a required surface. For an example, see Fig. 4.17.

When we use the Seifert algorithm instead of checkerboard coloring, we always obtain a bicollared disk-band representation. □

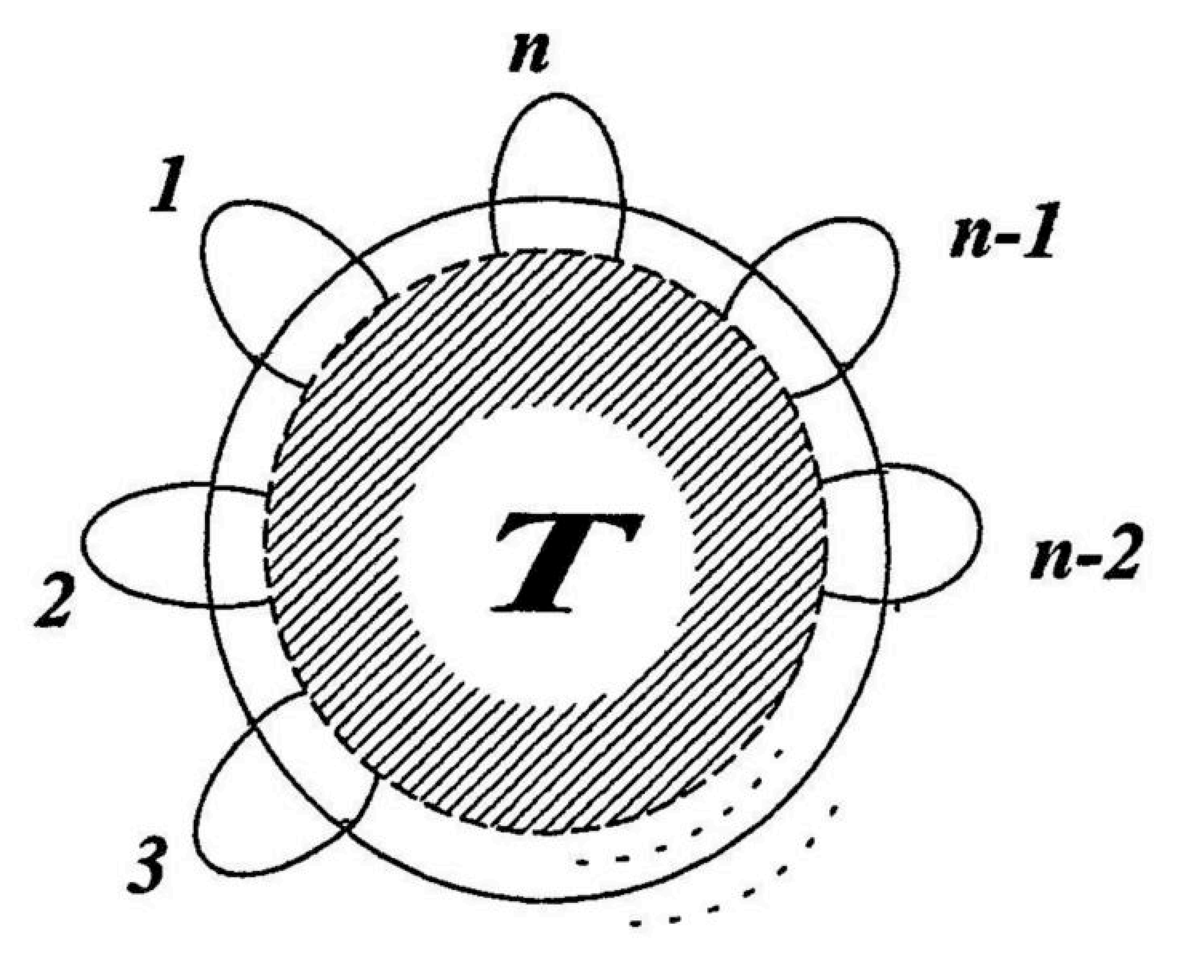


Fig. 4.15

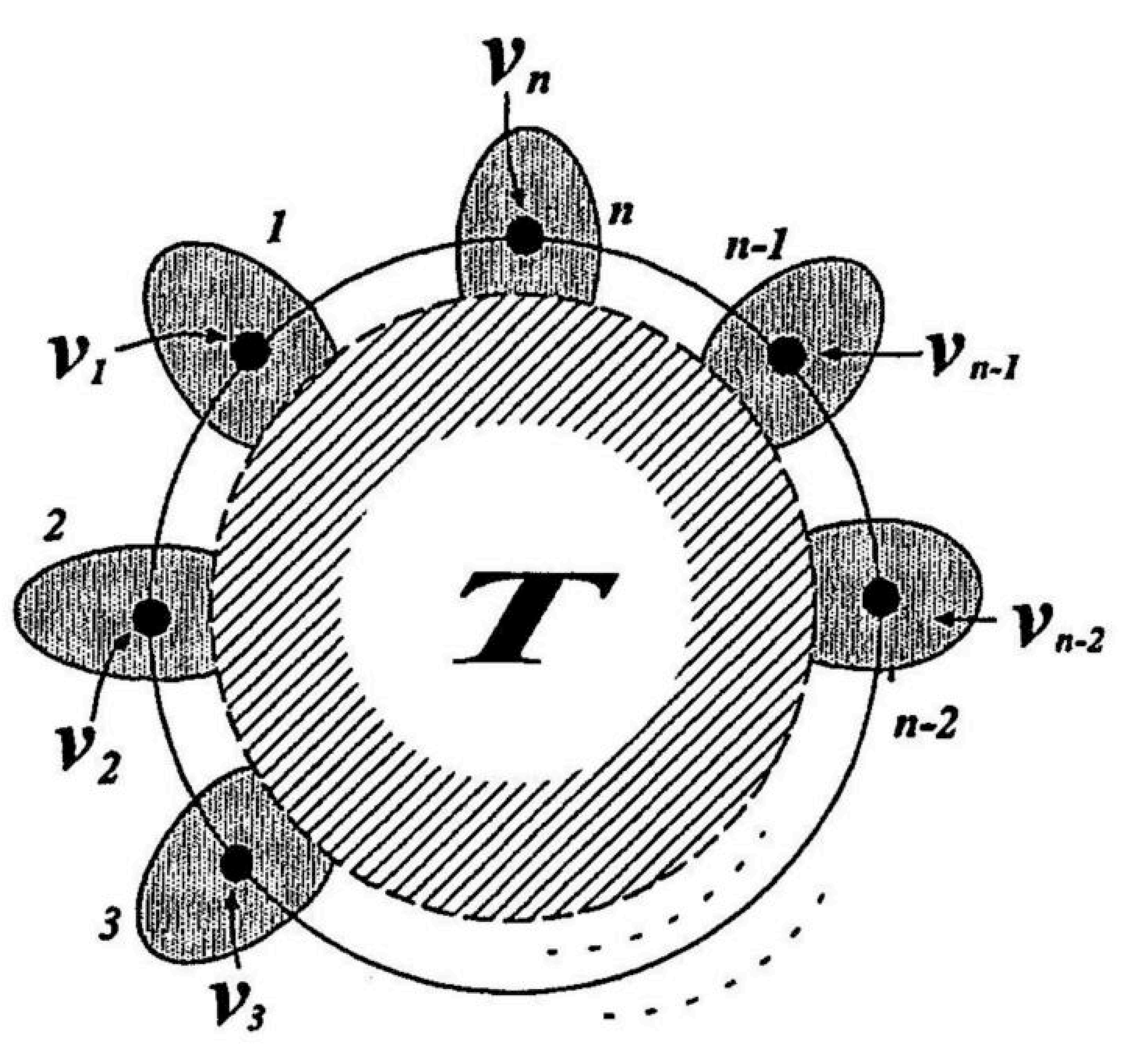


Fig. 4.16

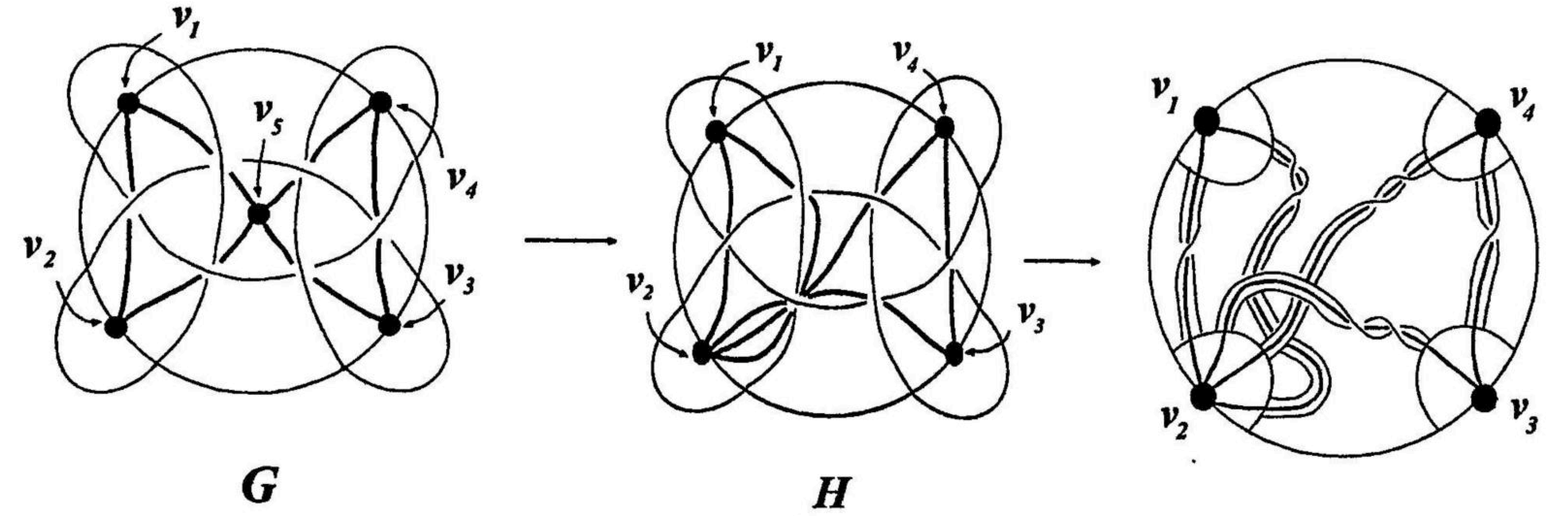


Fig. 4.17

4.2 Heegaard decompositions

In addition to the surgery presentation, it is also useful to have another presentation of a p-fold cyclic branched cover. Our construction leads straightforwardly to a Heegaard decomposition, that is a decomposition into a *compression body* (see [4, 6]) and a handlebody, of a p-fold cyclic branched cover.

Let F be a connected surface in B^3 bounded by an n-tangle T and n arcs in ∂B^3. The surface F is defined to be *free* if the exterior of F is homeomorphic to $(S_n \times I) \cup$(1-handles), where S_n is an n-punctured sphere and the all attaching points of the 1-handles are contained in $S_n \times \{0\}$. As we observed before, any connected surface has disk-band decomposition. A disk-band representation $\Omega(T_0; \{D_1, \cdots, D_n\}, \{b_1, \cdots, b_m\})$ is defined to be *free* if the surface $\bigcup_i D_i \cup \bigcup_j b_j$ is free. It is obvious that any disk-band representations constructed as in the proof of Proposition 4.1 are free. So any n-tangle has a free disk-band representation.

First we consider the case of two-fold branched covers. The following algorithm gives a Heegaard decomposition of $M^{(2)}(T)$. Start from a free disk-band representation of T. Then we have a surgery description $\Sigma(\varphi^{-1}(\bigcup_i b_i), H_0)$ of $M^{(2)}(T)$ as in Fig. 4.3. Let $\alpha_1, \alpha_2, ..., \alpha_{2m-n}$ be the $2m-n$ arcs which are the connected components of $T_0 - \bigcup_j b_j$ contained in the interior of B^3 as in Fig.4.18. Then the complement of $\varphi^{-1}(\bigcup_i b_i \bigcup_l \alpha_l)$ is a compression body. Then $M^{(2)}(T)$ is obtained from the compression body by gluing the handlebody as follows; (i) adding meridian disks of the arcs, and (ii) filling the rest according to the surgery description. For the tangle T_1 in Example 4.1, a Heegaard decomposition of $M^{(2)}(T_1)$ is given in Fig. 4.19.

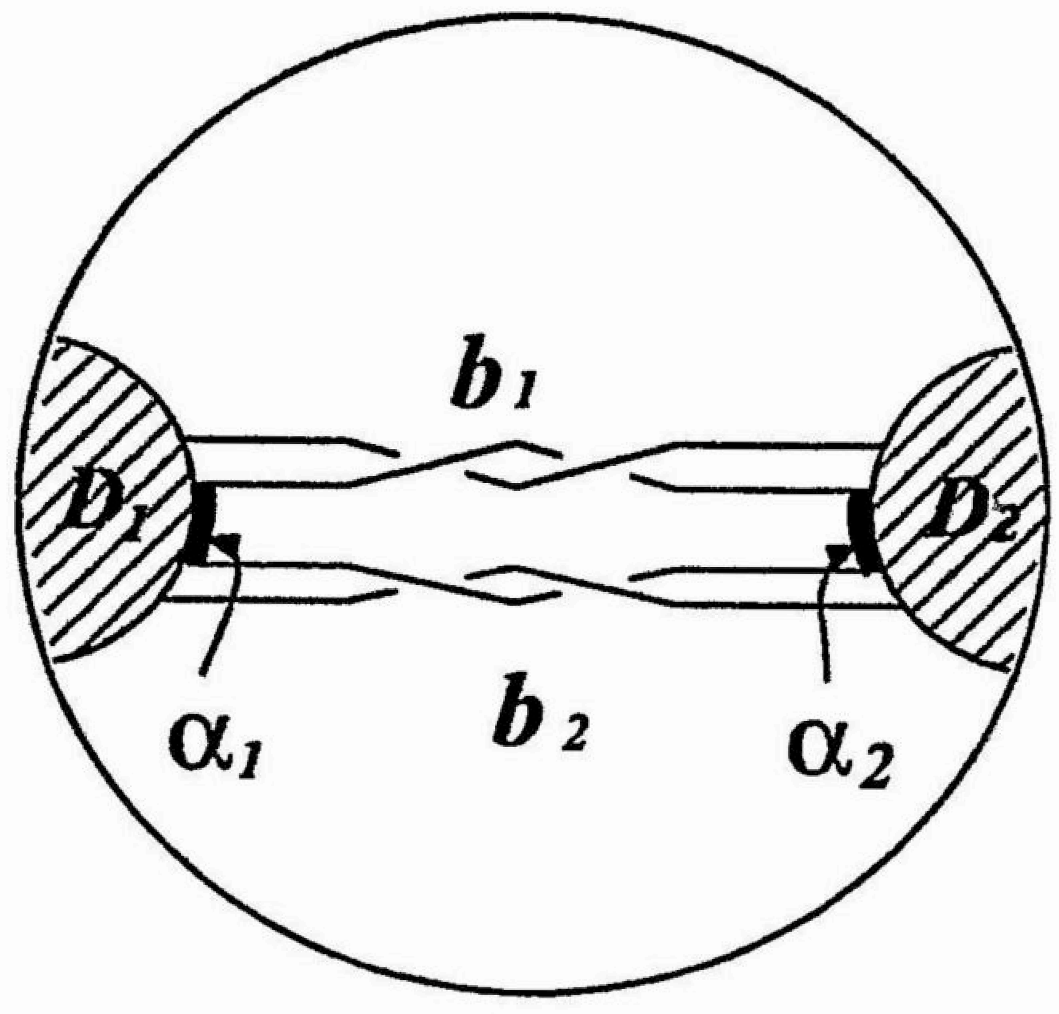
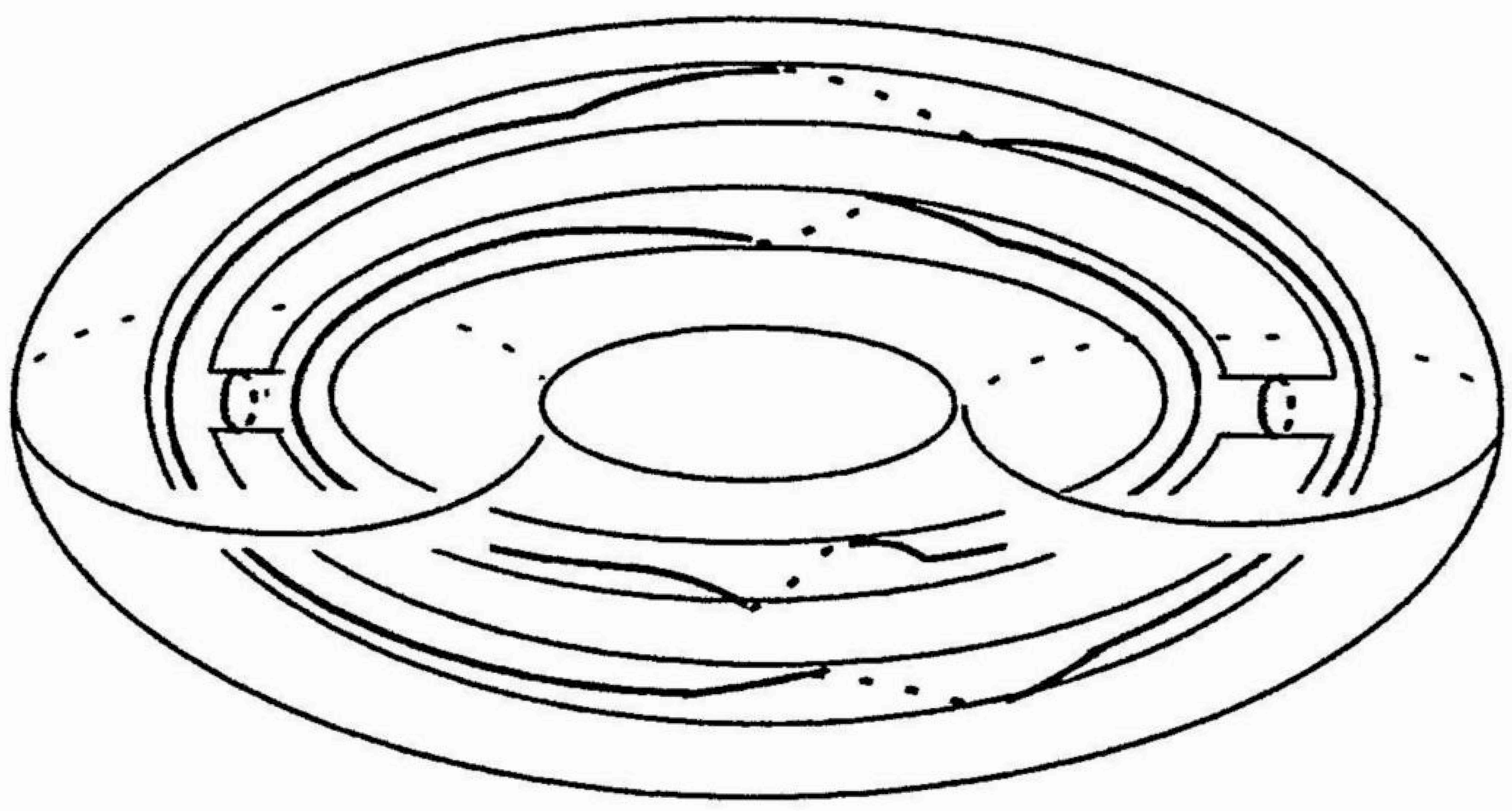


Fig. 4.18



Heegaard decomposition of $M^{(2)}(T_1)$

Fig. 4.19

A similar method gives a Heegaard decomposition of a p-fold cyclic branched cover. Our construction is a modification of the construction in the proof of Theorem 4.2. The handlebody part of the decomposition is obtained from the handlebodies $\bigcup_j \varphi^{-1}(b_j \times [-1,1])$ by connecting them using $2m-n$ "tubes" along $\varphi^{-1}(T_0)$. We get genus $mp-n+1$ handlebody.

From the observation above it follows that:

Theorem 4.3 *Let $\Omega(T_0; \{D_1, \cdots, D_n\}, \{b_1, \cdots, b_m\})$ be a free, bicollared disk-band representation of an n-tangle T in B^3. Let $\varphi : H_0 \to B^3$ be the p-fold cyclic branched cover of branched along T_0. Let $\alpha_1, \alpha_2, ..., \alpha_{2m-n}$ be the $2m - n$ arcs which are the connected components of $T_0 - \bigcup_j b_j$ contained in the interior of B^3. Then the following holds.*

(a) *The complement $W = \overline{H_0 - \varphi^{-1}(\bigcup_j b_j \times [-1,1] \cup \bigcup_l N(\alpha_l))}$ is a compression body, where $N(\alpha_l)$ is the tubular neighborhood of α_l in B^3.*

(b) *The p-fold cyclic branched cover of B^3 branched along T has a Heegaard decomposition into the compression body W and a genus $mp - n + 1$ handlebody.*

(c) *The gluing map is given by the curves $c_{j,k}$ $(j = 1, 2, ..., m,\ k = 1, 2, ..., p-1)$ and m_l $(l = 1, 2, ..., 2m - n)$ in ∂W, where $c_{j,k}$ is the core of the annulus $b^+_{j,k} \cup b^-_{j,k+1}$ in Theorem 4.2 and m_l is the meridian curve of $\varphi^{-1}(N(\alpha_l))$.* □

In the theorem above, the assumption that a disk-band representation is bicollared is not necessary in the case that $p = 2$. The curves $c_{j,k}$ $(j = 1, 2, ..., m,\ k = 1, 2, ..., p - 1)$, m_l $(l = 1, 2, ..., 2m - n)$ are essential, $mp - n + 1$ of them are non-separating and $m - 1$ curves, m_l's, are separating.

Remark 4.1 *Since the surfaces given in the proof of Proposition 4.1 are connected and free, we can use them to find Heegaard decompositions of branched cyclic covers. Let c denote the crossing number of a connected diagram $D(T_0 \cup T)$, b the number of the black regions and s the number of the Seifert circles of $D(T_0 \cup T)$. Then we have a Heegaard decomposition of $M^{(2)}(T)$ (resp. $M^{(p)}(T)$) of the genus $n + 2c - 2b + 1$ (resp. $p(n + c - s) - n + 1$).*

4.3 Rotation and double branched covers of certain tangles

As an application of this construction, we examine the branched covering space of rotant links. In general, it is not true that a rotation preserves the first homology group of the double branched cover S^3 branched along a link. Necessary conditions for preservation of the homology group are given in [9]. However, if we assume that a pair of oriented rotants with "special" disk-band representation, then we conclude that the first homology groups of the double branched covers of S^3 branched along a pair of rotants are isotopic. In fact we have the following theorem.

Theorem 4.4 *Let L_1 and L_2 be a pair of rotants of any order n. Assume that n-rotors R_k of L_k are in the disk-band representations $\Omega(T_0; \{D_1, ..., D_n\}, \{b_{k1}, ..., b_{kn}\})$ $(k = 1, 2)$ which have also n-rotational symmetries. Let $M^{(2)}_{L_k}$ $(k = 1, 2)$ be the double branched cover of S^3 branched along a link L_k. Then the first homology groups of $M^{(2)}_{L_1}$ and $M^{(2)}_{L_2}$ are isomorphic.*

Note that the numbers of the elements of $\{D_1, \cdots, D_n\}$ and $\{b_{k_1}, \cdots, b_{k_n}\}$ are the same.

Proof of Theorem 4.4 Let B^3 be the 3-ball such that $B^3 \cap L_k = \Omega(T_0; \{D_1, ..., D_n\}, \{b_{k1}, ..., b_{kn}\})$ $(k = 1, 2)$ and $B^3_0 = B^3 - \mathrm{int}N(D_1 \cup \cdots \cup D_n)$, where $N(D_1 \cup \cdots \cup D_n)$ is a regular neighborhood of $D_1 \cup \cdots \cup D_n$ in B^3. There are compact, connected, possibly non-orientable surfaces F_k $(k = 1, 2)$ in S^3 such that $F_k \cap B^3 = D_1 \cup \cdots \cup D_n \cup b_{k1} \cup \cdots \cup b_{kn}$. Let v_i be a point in $D_i \cap \partial B^3$ $(i = 1, ..., n)$. Let G_k be a spine of F_k with vertex set $\{v_1, ..., v_n\}$ such that $G_k \cap B^3$ is a spine of $D_1 \cup \cdots \cup D_n \cup b_{k1} \cup \cdots \cup b_{kn}$. Let $T_k \subset S^3 - \mathrm{int}B^3$ be a spanning tree of G_k and G_k/T_k a spine obtained from G_k by contracting T_k into a point v. We may assume that $N(G_k/T_k) \cap F_k$ consists of a disk D_{k0} containing v and mutually disjoint disks $b'_{k1}, ..., b'_{km}$ such that $b'_{ki} \cap D_{k0} = \partial b'_{ki} \cap \partial D_{k0}$ are two disjoint arcs in $\partial b'_{ki}$ $(i = 1, ..., m)$, $(D_{k0} \cup b'_{k1} \cup \cdots \cup b'_{km}) \cap B^3_0 =$

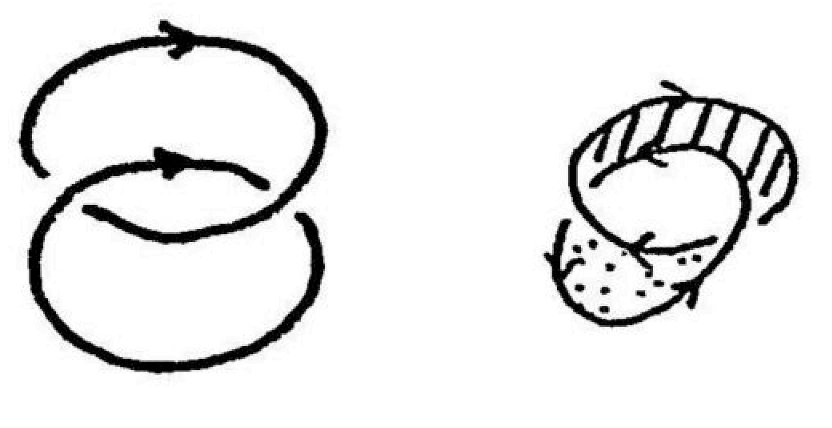
$(b_{k1} \cup \cdots \cup b_{kn}) \cap B_0^3$, and that $(D_{10} \cup b'_{11} \cup \cdots \cup b'_{1m}) - B_0^3 = (D_{20} \cup b'_{21} \cup \cdots \cup b'_{2m}) - B_0^3$. Let $\varphi : S^3 \longrightarrow S^3$ be the double branched cover branched along ∂D_{k0}. Then $M_{L_k}^{(2)}$ is obtained from S^3 by surgery along a framed link $\varphi^{-1}(b'_{k1} \cup \cdots \cup b'_{km})$. Note that $\varphi^{-1}((b'_{k1} \cup \cdots \cup b'_{km}) \cap B_0^3) = \varphi^{-1}((b_{k1} \cup \cdots \cup b_{kn}) \cap B_0^3)$ are two n-rotors and $\varphi^{-1}((b'_{11} \cup \cdots \cup b'_{1m}) - B_0^3) = \varphi^{-1}((b'_{21} \cup \cdots \cup b'_{2m}) - B_0^3)$. Since each $\varphi^{-1}(b'_{ki})$ is a component of $\varphi^{-1}(b'_{k1} \cup \cdots \cup b'_{km})$, it is not hard to see that there is a blackboard framed, oriented link $c_{k1} \cup \cdots \cup c_{km}$ such that each c_{ki} corresponds to b'_{ki} and the both component of $(c_{k1} \cup \cdots \cup c_{km}) \cap \varphi^{-1}(B_0^3)$ are oriented n-rotors. So $c_{21} \cup \cdots \cup c_{2m}$ is obtained from $c_{11} \cup \cdots \cup c_{1m}$ by twice oriented n-rotations. By Theorem 3.1 (iii), the linking matrices of $c_{11} \cup \cdots \cup c_{1m}$ and $c_{21} \cup \cdots \cup c_{2m}$ coincide. Since the linking matrix of $c_{k1} \cup \cdots \cup c_{km}$ is a relation matrix of the first homology group of $M_{L_k}^{(2)}$, we have the conclusion. □

A Appendix : Exhibition of minimal genus Seifert surfaces of prime links

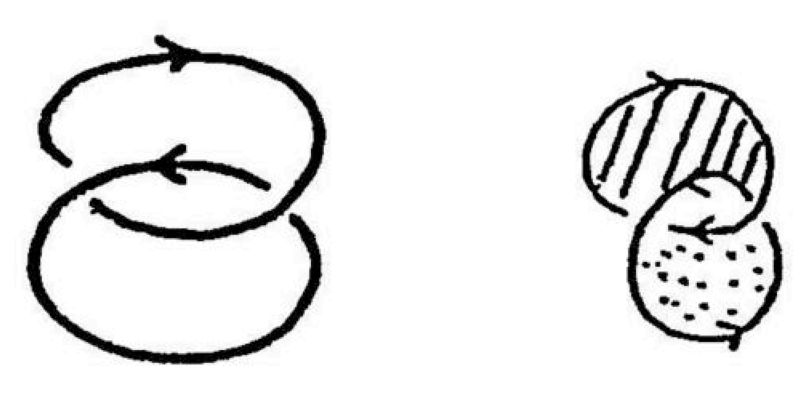
We exhibit minimal genus Seifert surfaces of prime n-component links ($n = 2, 3, 4$) with up to 9 crossings (for the notation we refer [34, Appendix C]). We determine the genera of links by applying product decompositions and disk decompositions come from the sutured manifold theory ([11, Theorem 4.1]).

A minimal genus Seifert surface is displayed on the right-hand side of each link diagram. We assign orientations to each link diagram in the table of [34] and indicate the orientation classes of the 2-component links (resp. 3-component links and 4-component links) by the number ① and ② (resp. ⓜ ($1 \leq m \leq 4$) and ⓝ ($1 \leq n \leq 8$)).

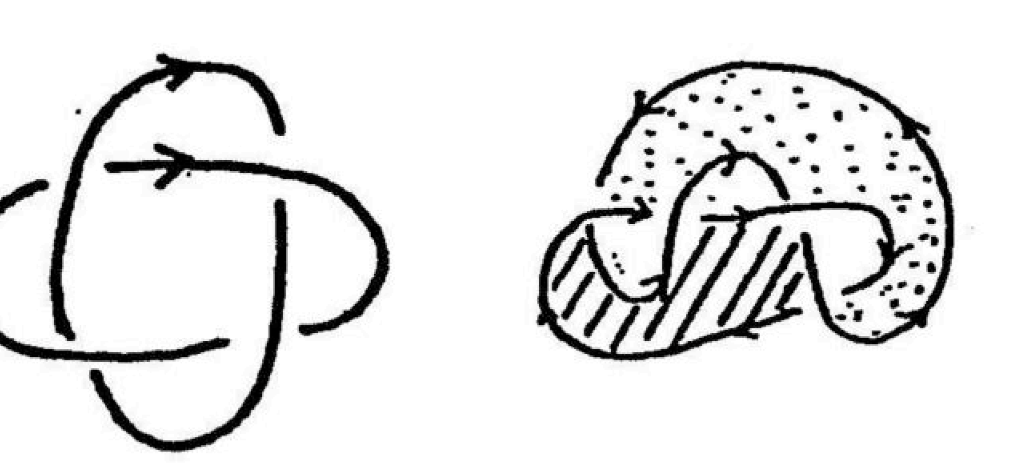
A.1 2-component prime links



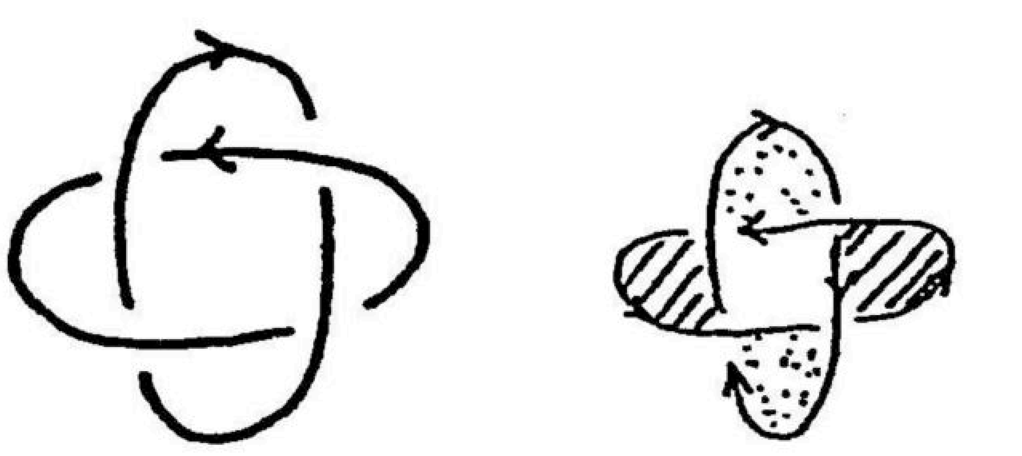
$2^2_1$①



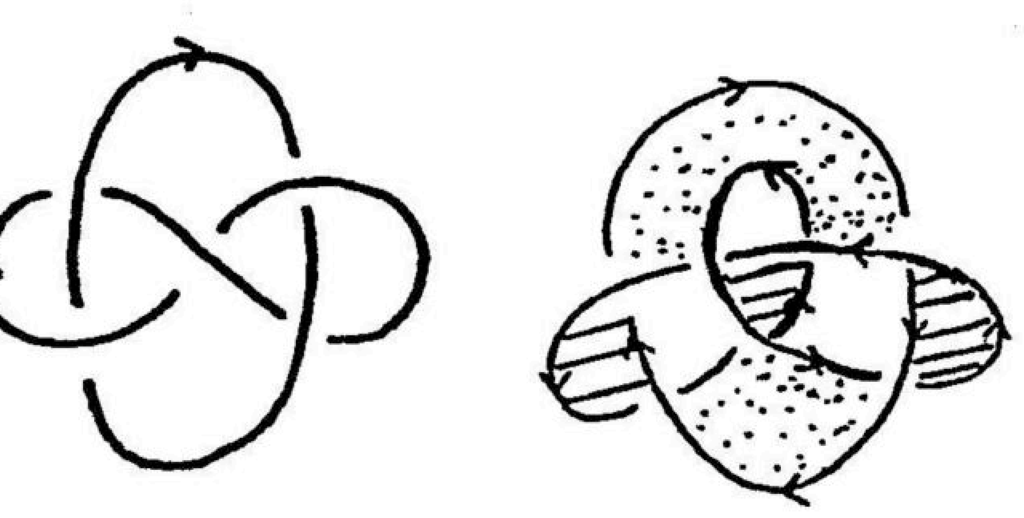
$2^2_1$②



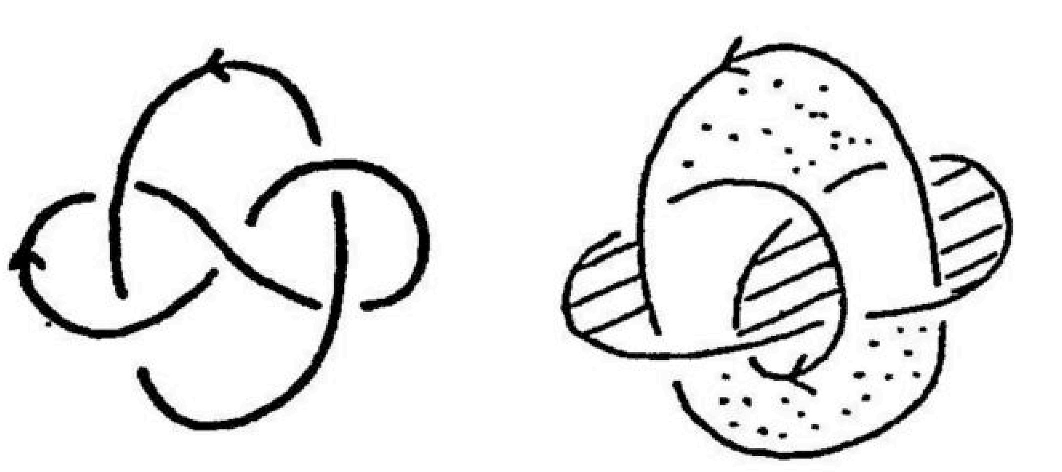
$4^2_1$①



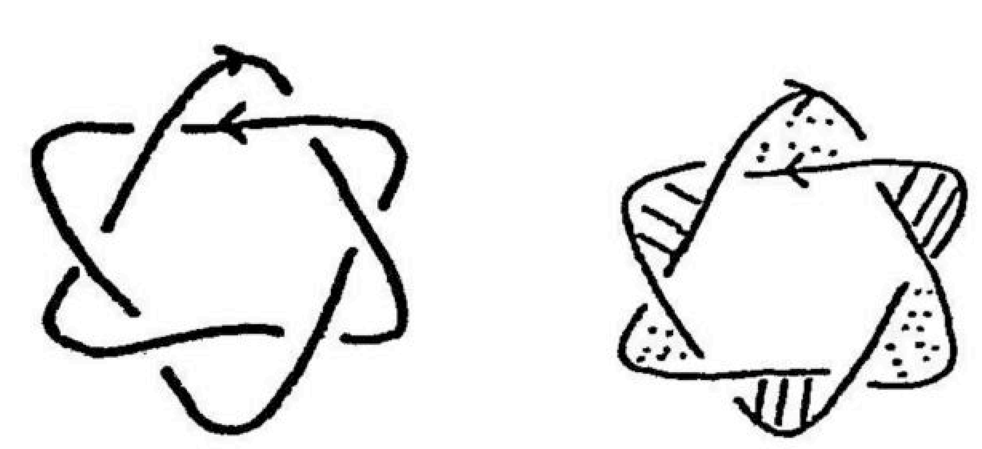
$4^2_1$②



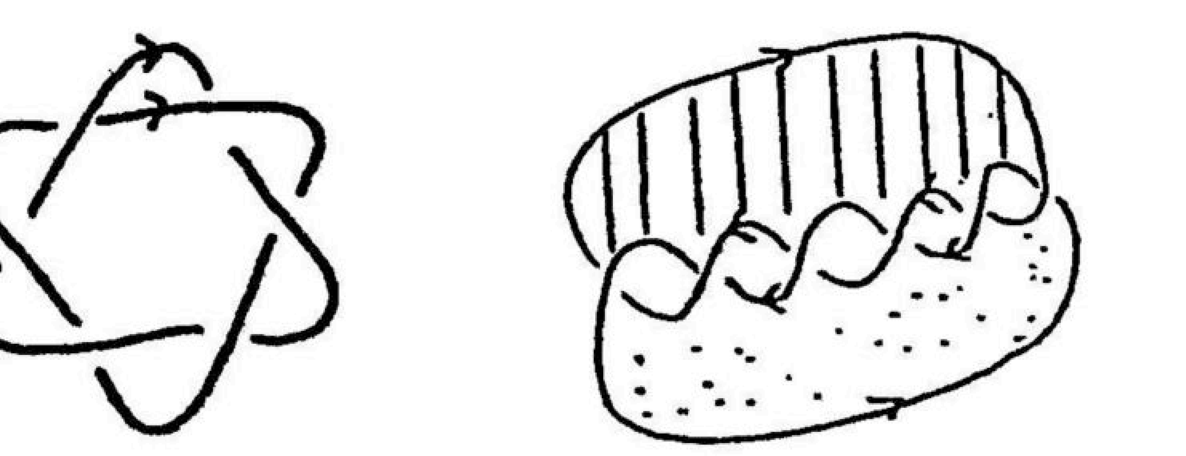
$5^2_1$①



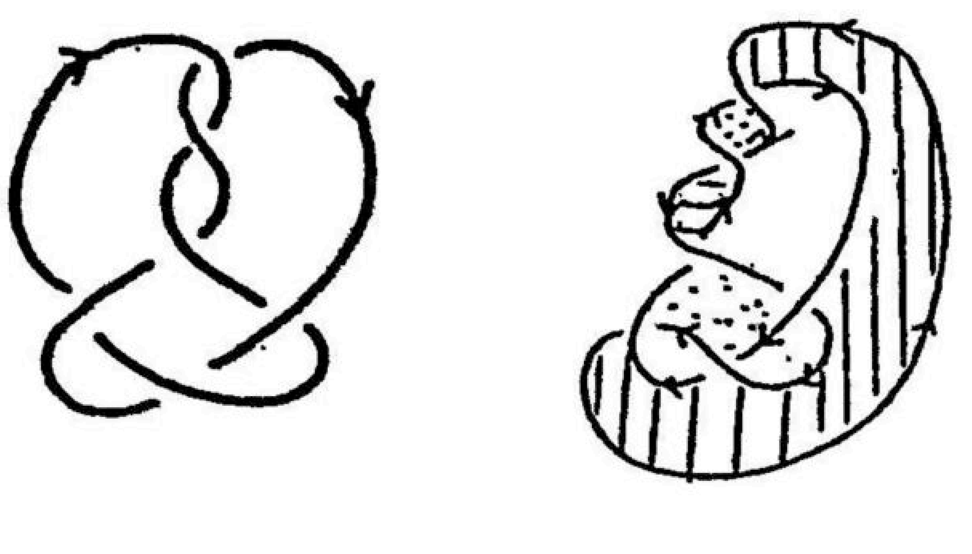
$5^2_1$②



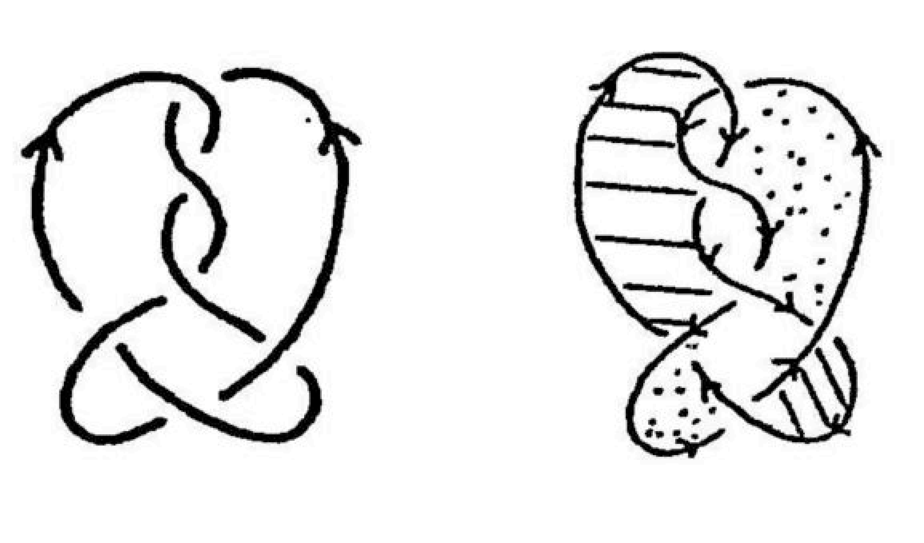
$6^2_1$①



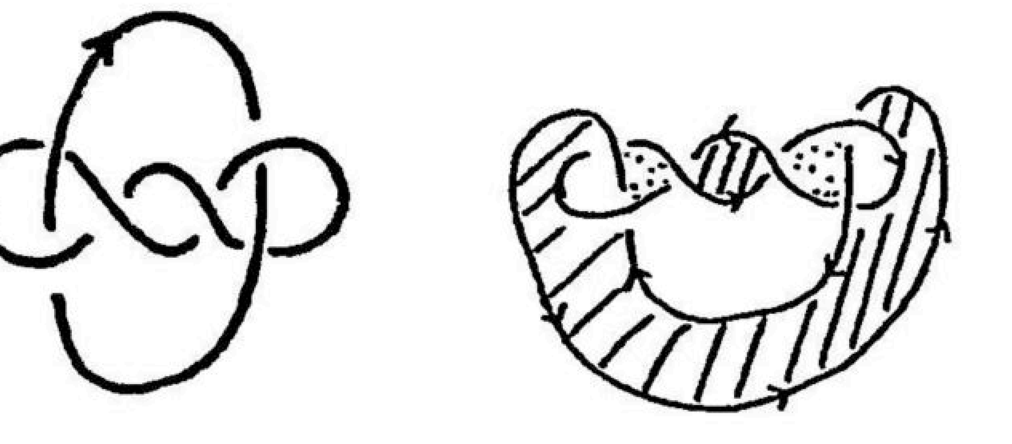
$6^2_1$②



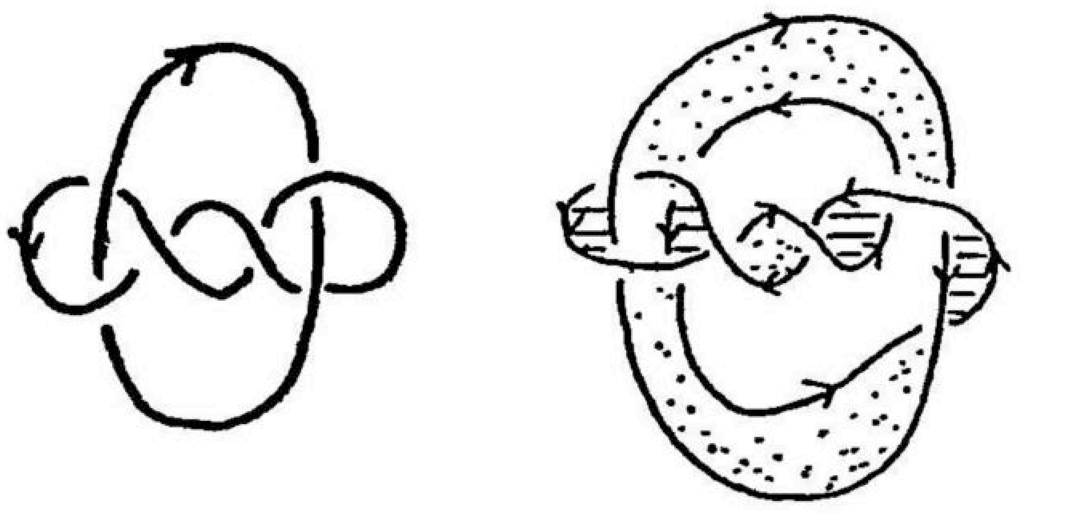
$6^2_2$①



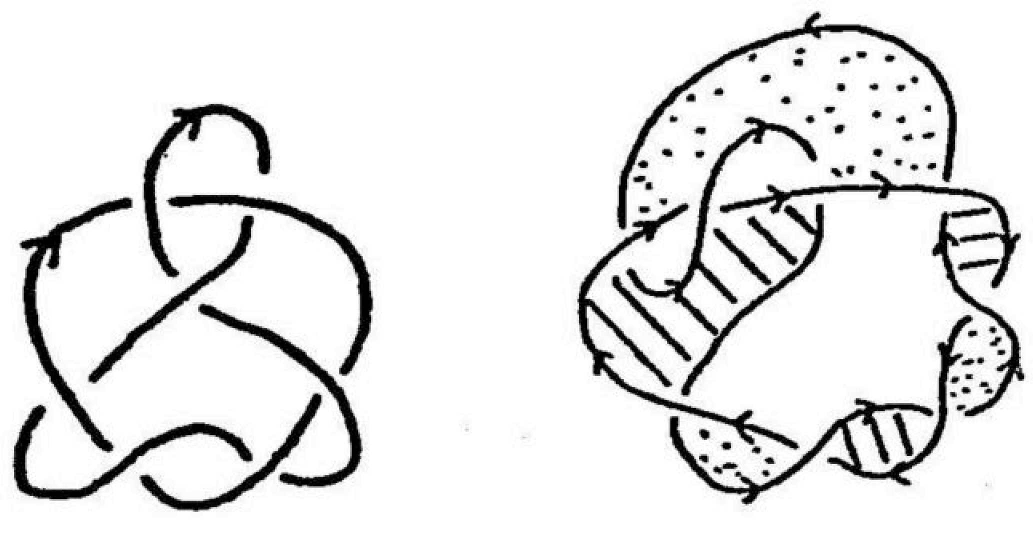
$6^2_2$②



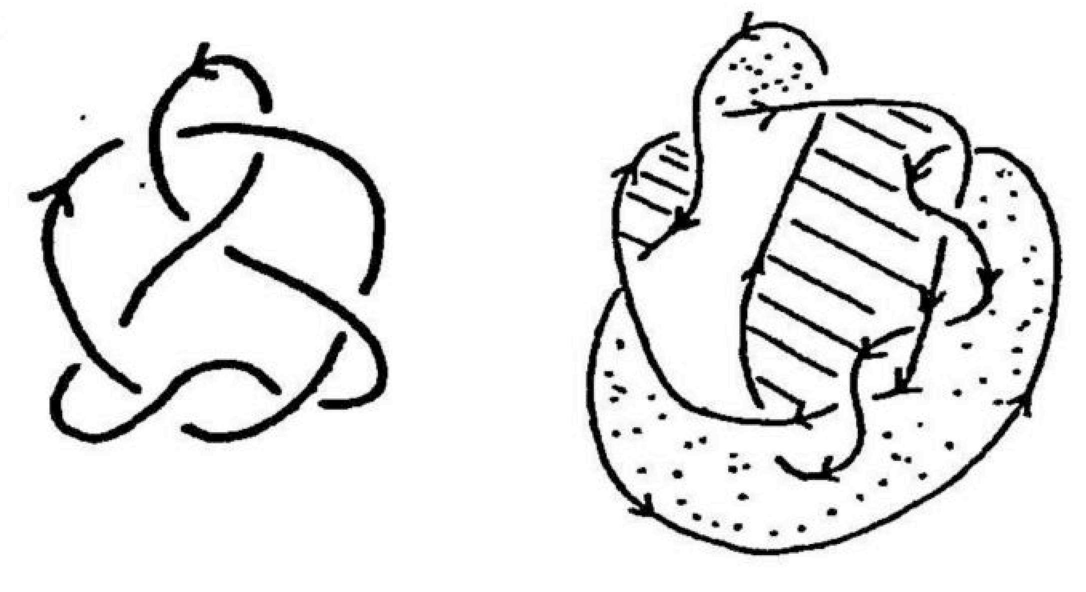
$6^2_3$①



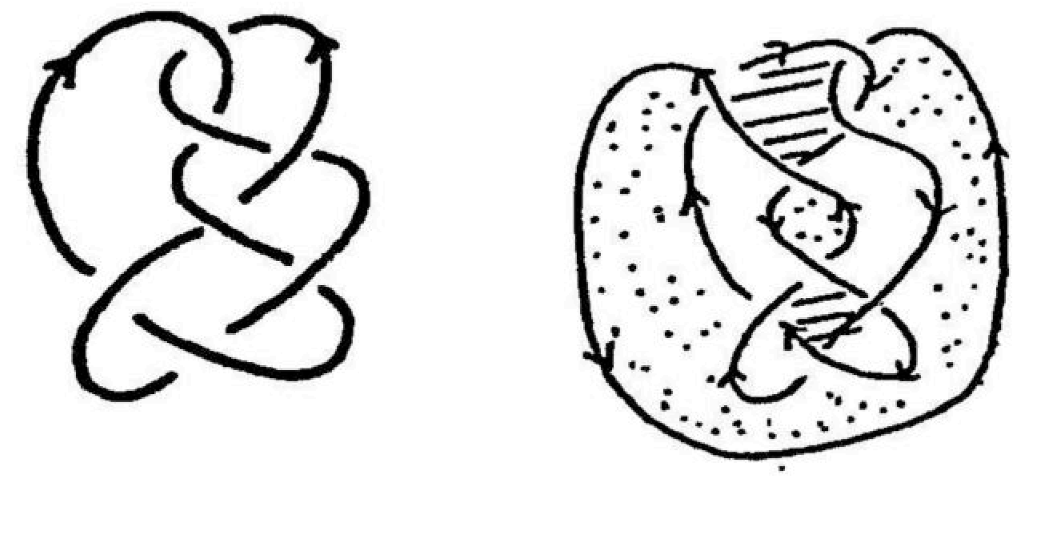
$6^2_3$②



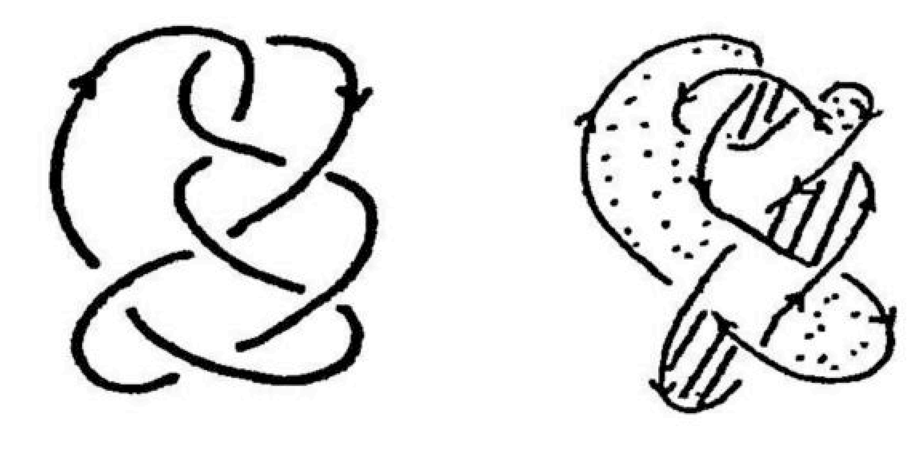
$7^2_1$①



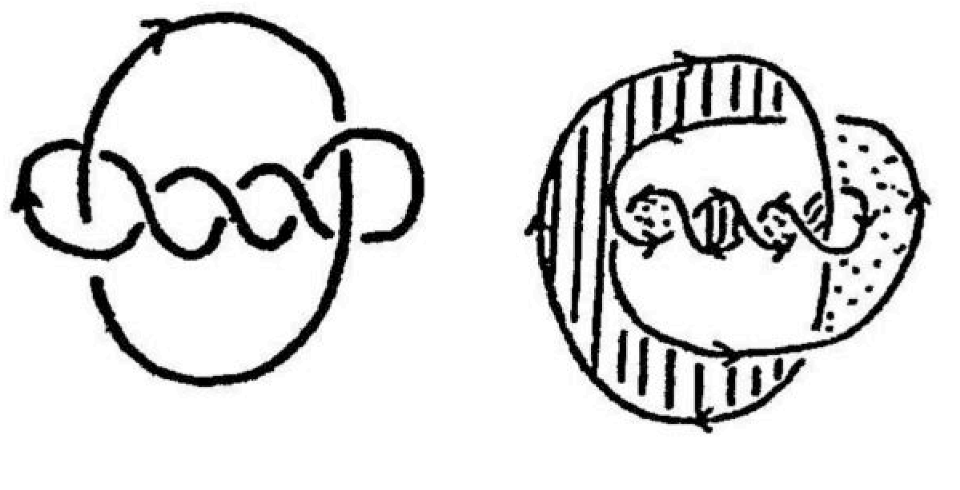
$7^2_1$②



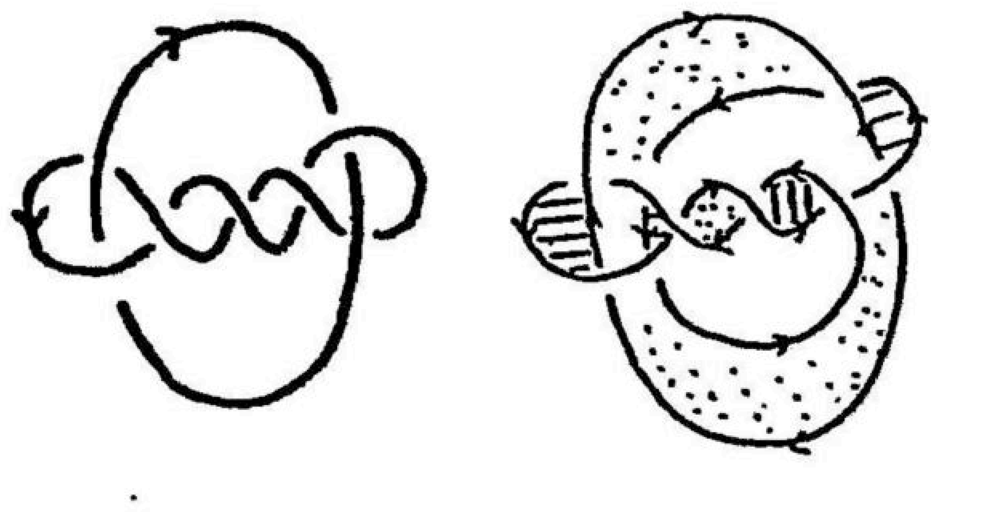
$7^2_2$①



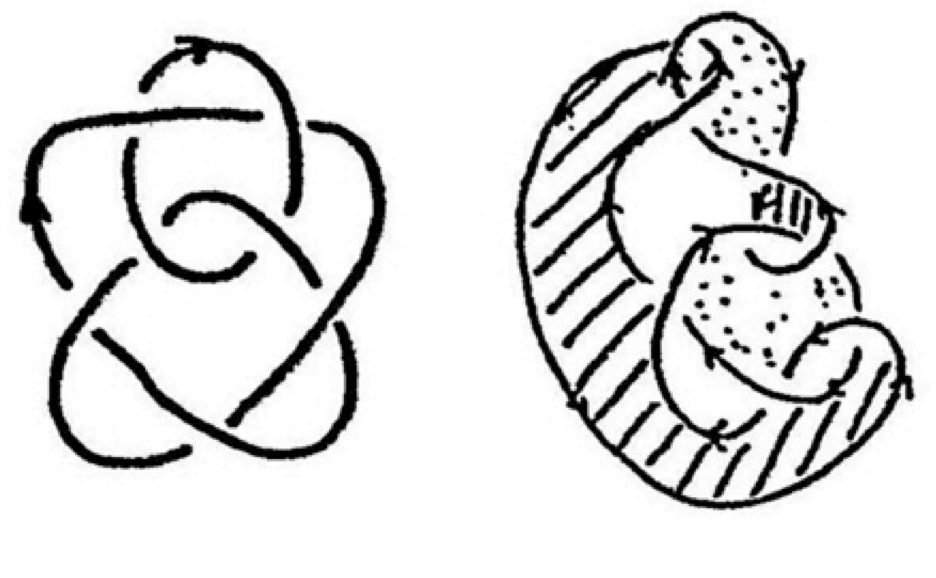
$7^2_2$②



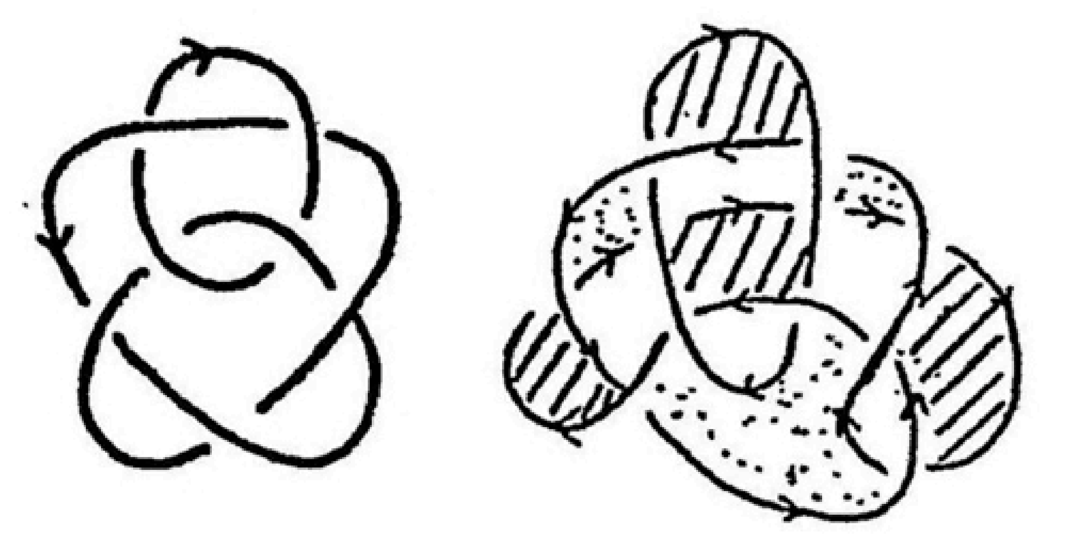
$7^2_3$①



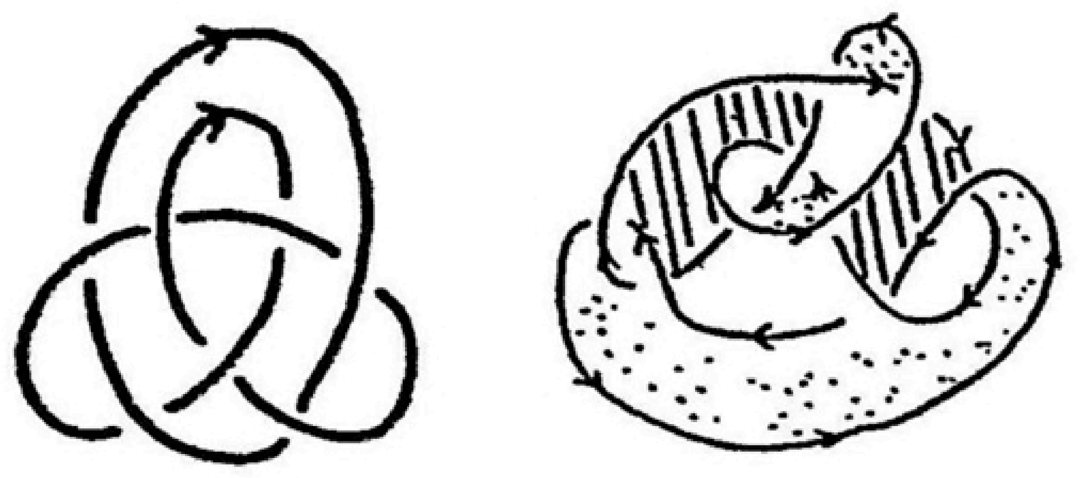
$7^2_3$②



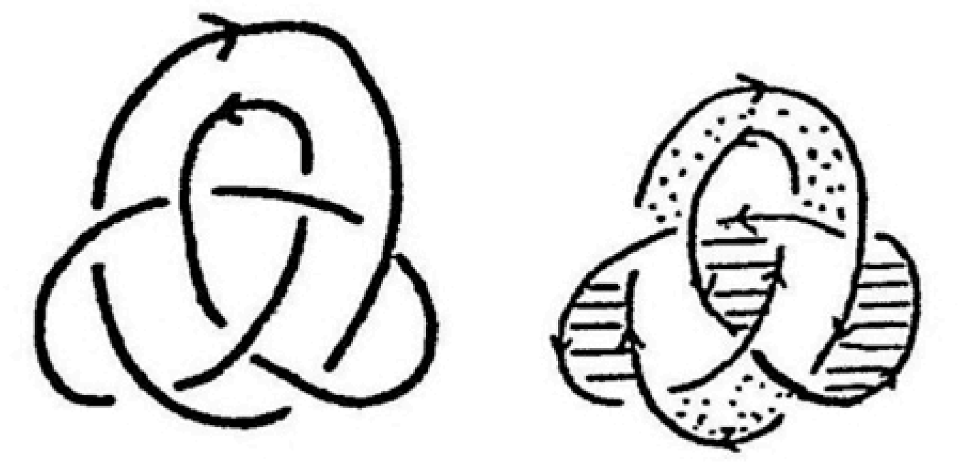
$7^2_4$①



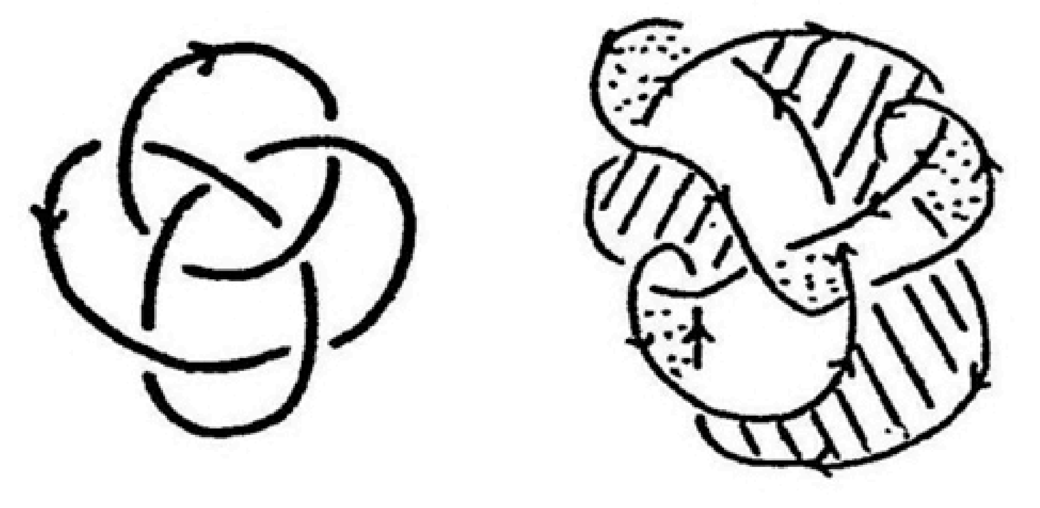
$7^2_4$②



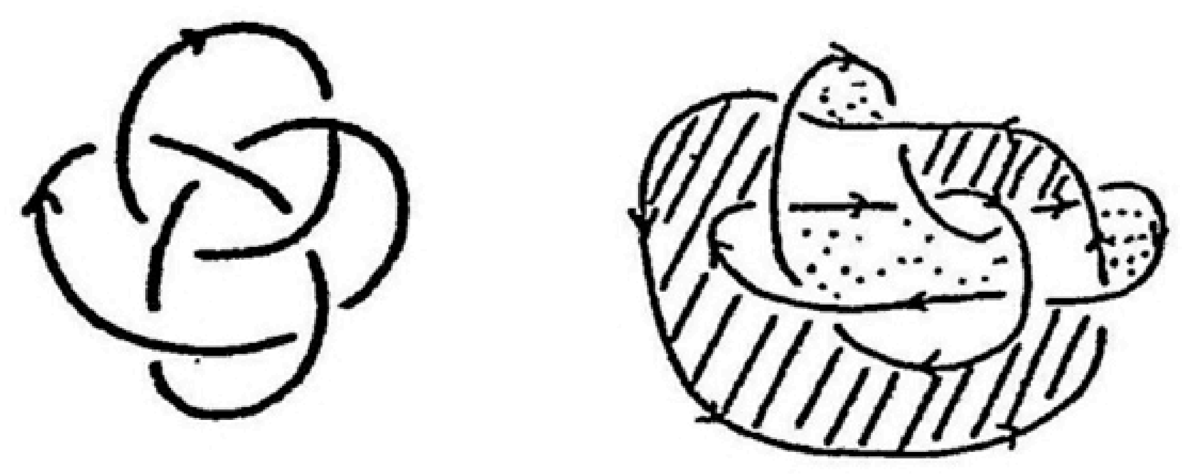
$7^2_5$①



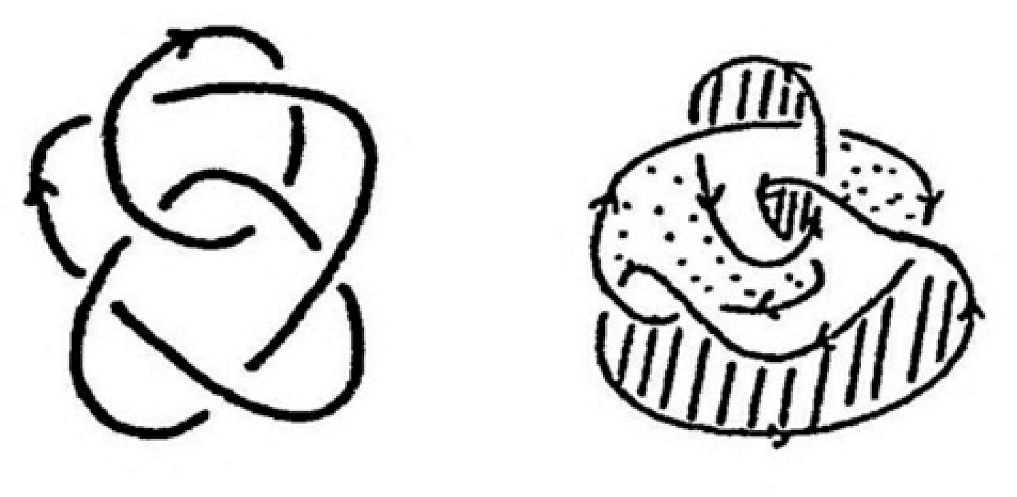
$7^2_5$②



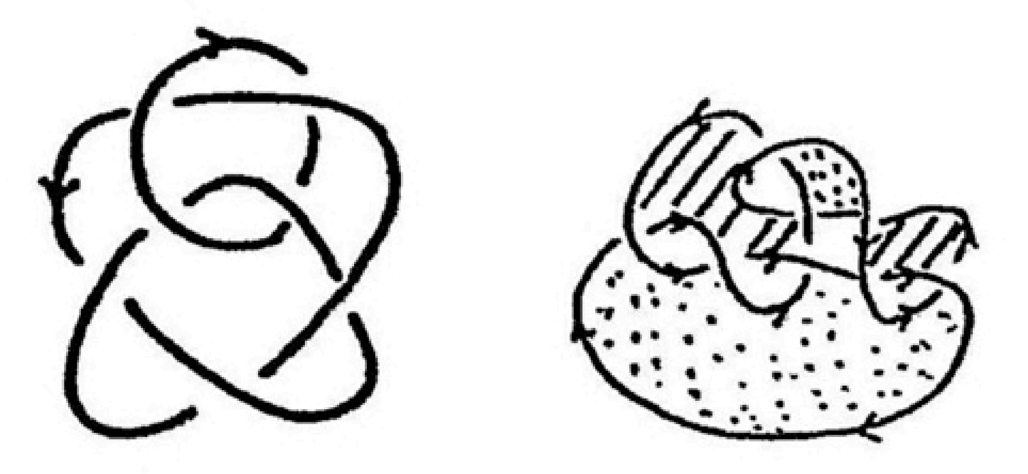
$7^2_6$①



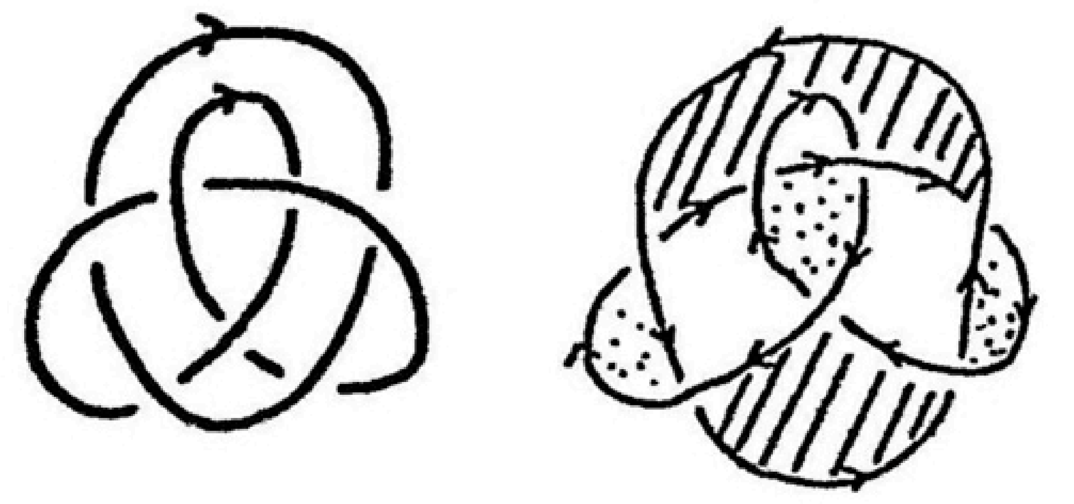
$7^2_6$②



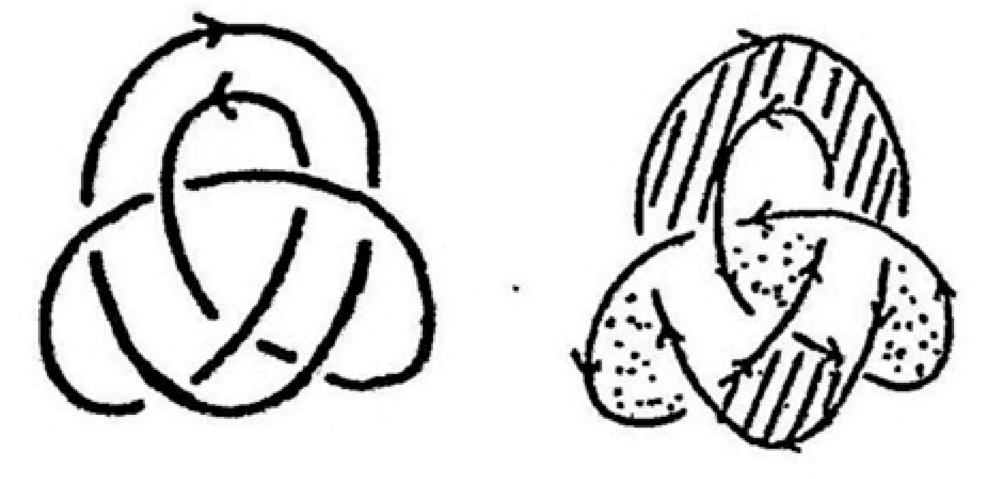
$7^2_7$①



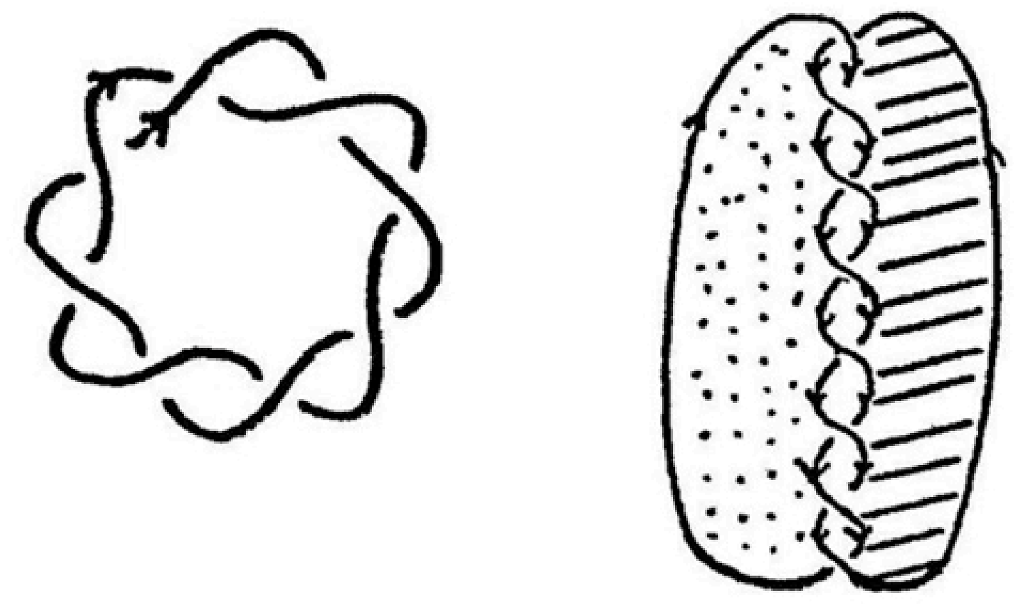
$7^2_7$②



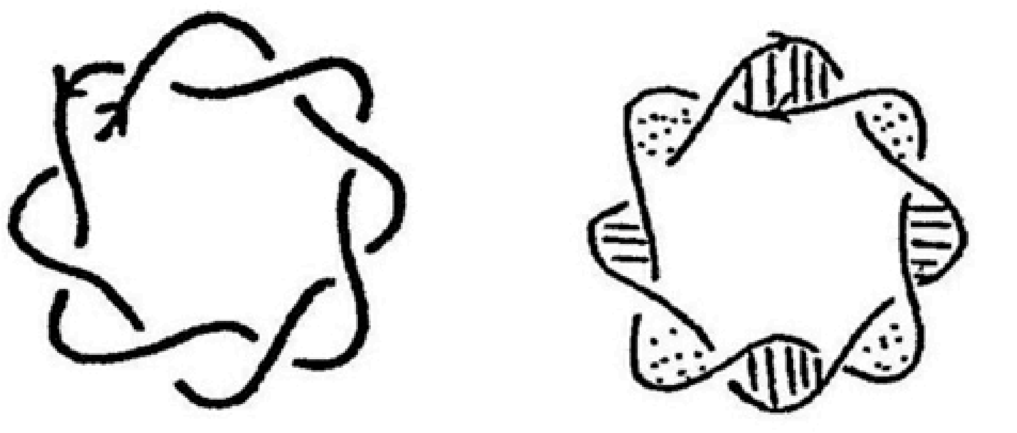
$7^2_8$①



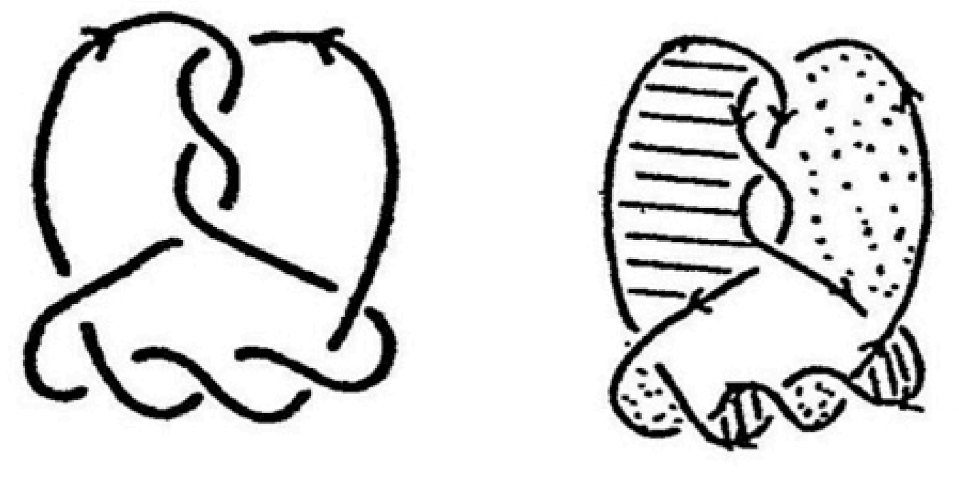
$7^2_8$②



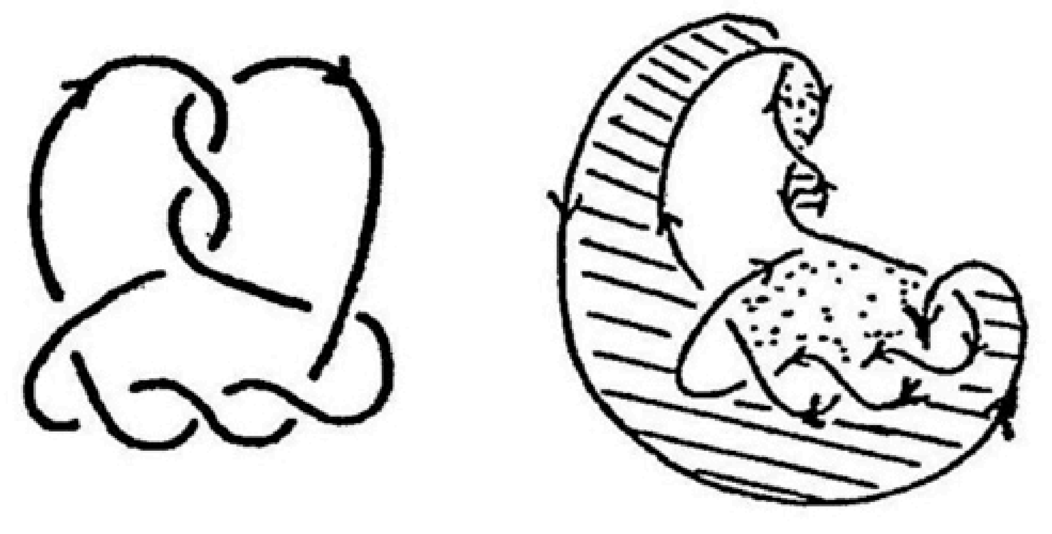
$8^2_1$①



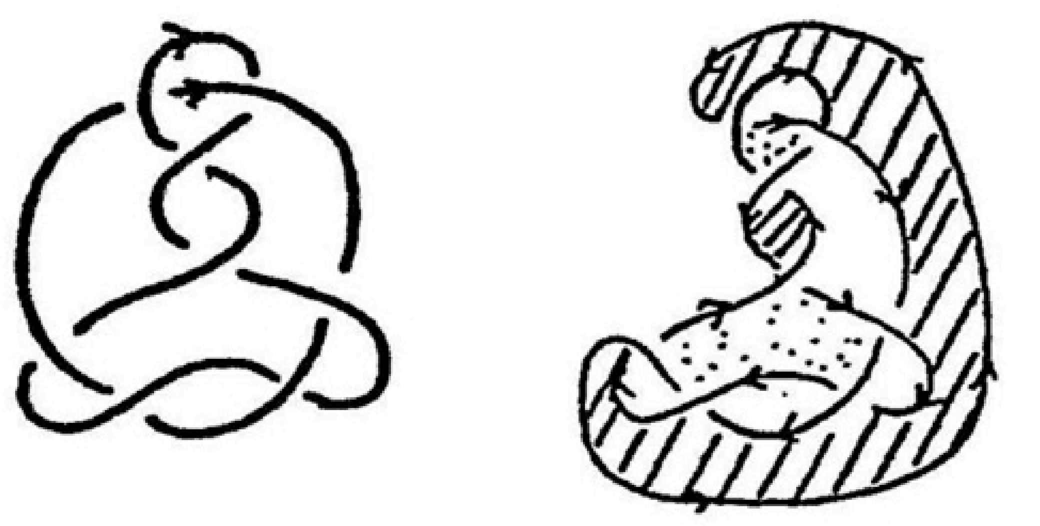
$8^2_1$②



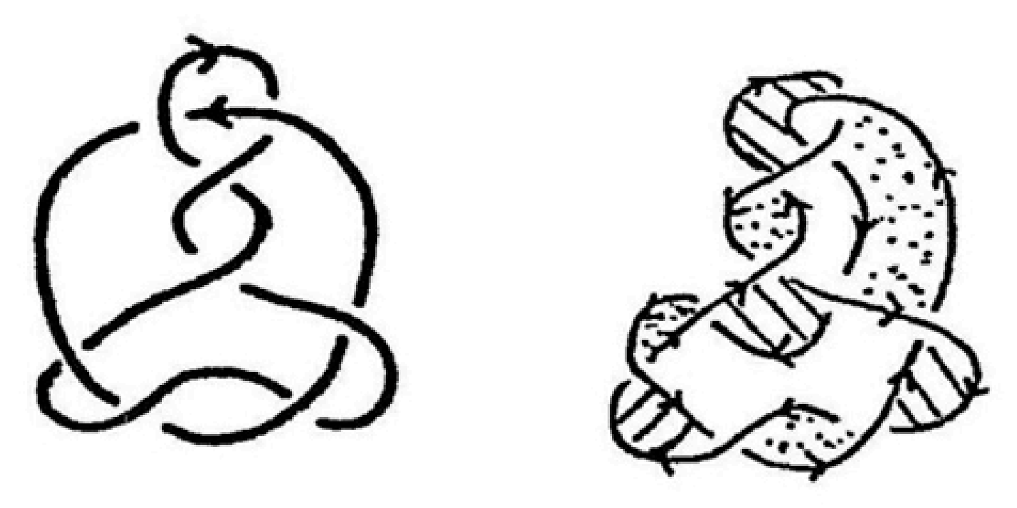
$8^2_2$①



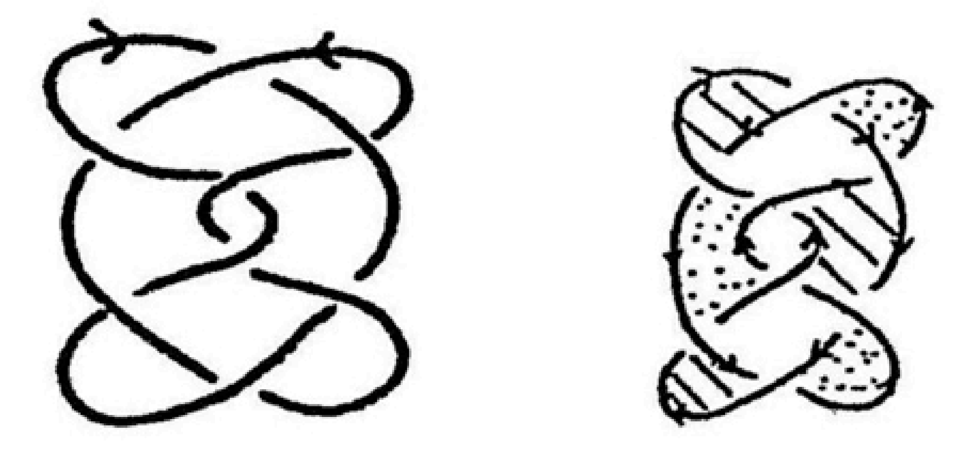
$8^2_2$②



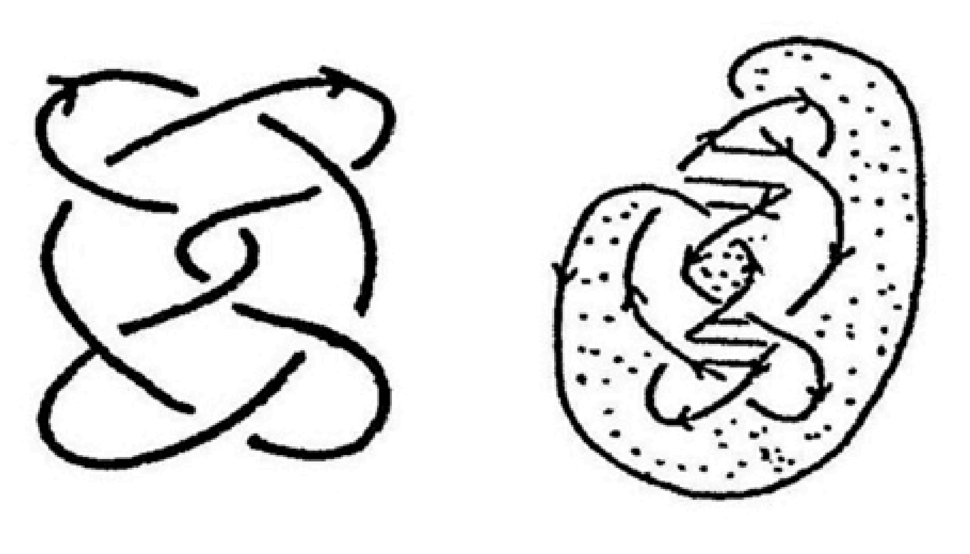
$8^2_3$①



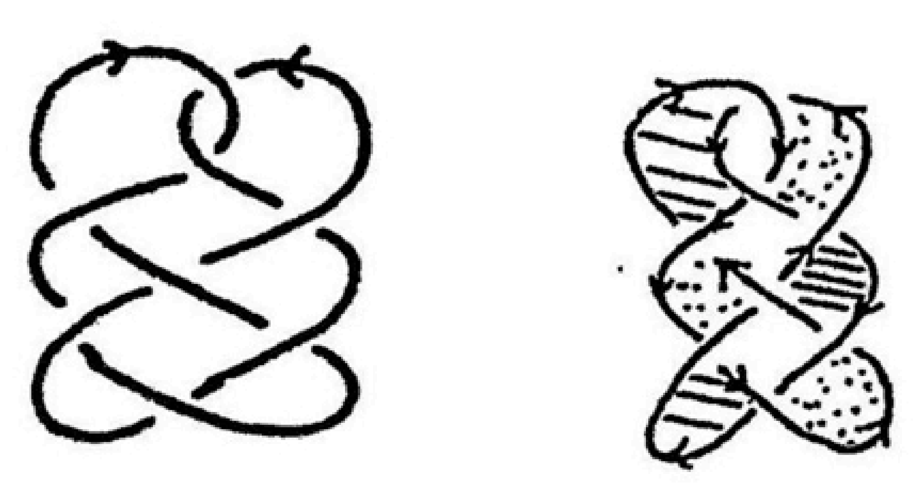
$8^2_3$②



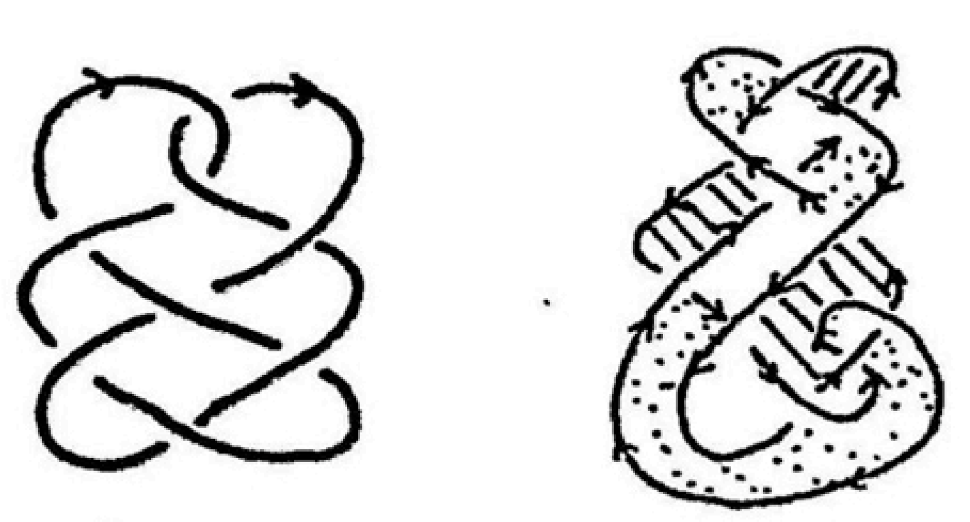
$8^2_4$①



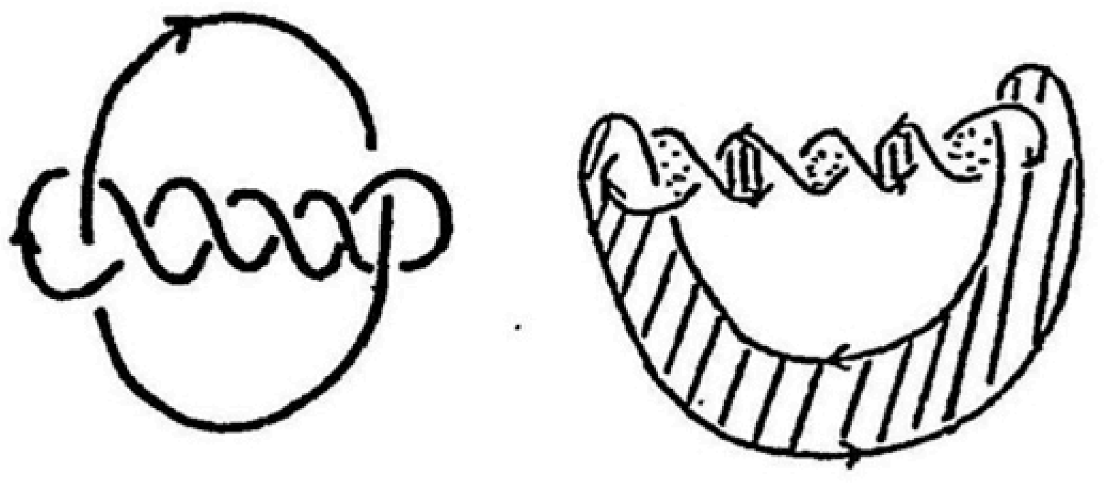
$8^2_4$②



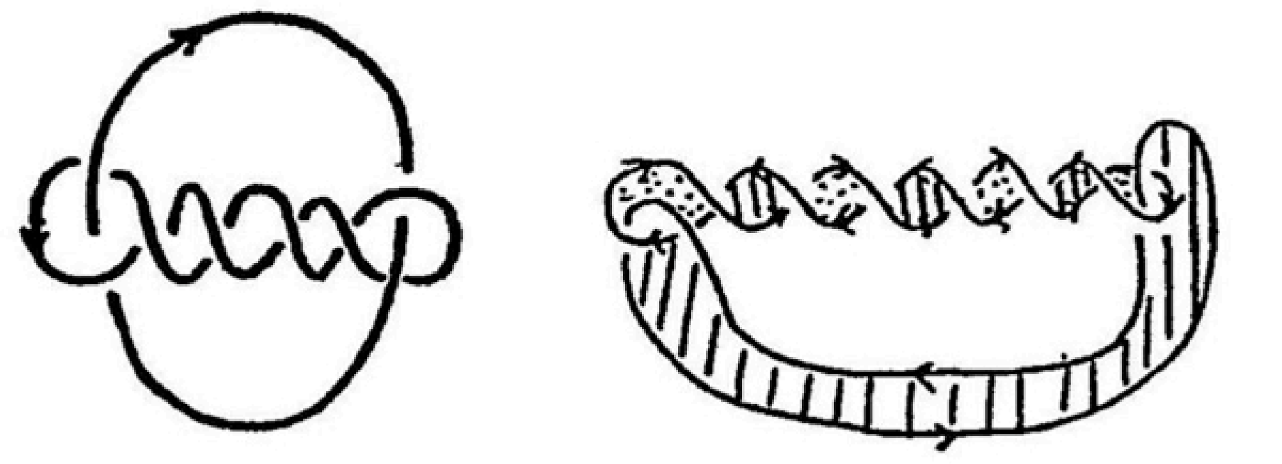
$8^2_5$①



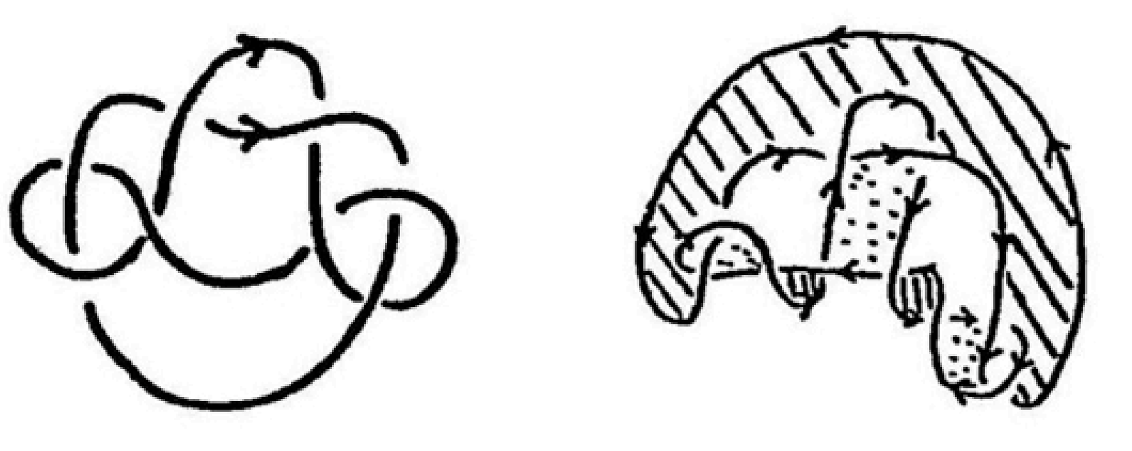
$8^2_5$②



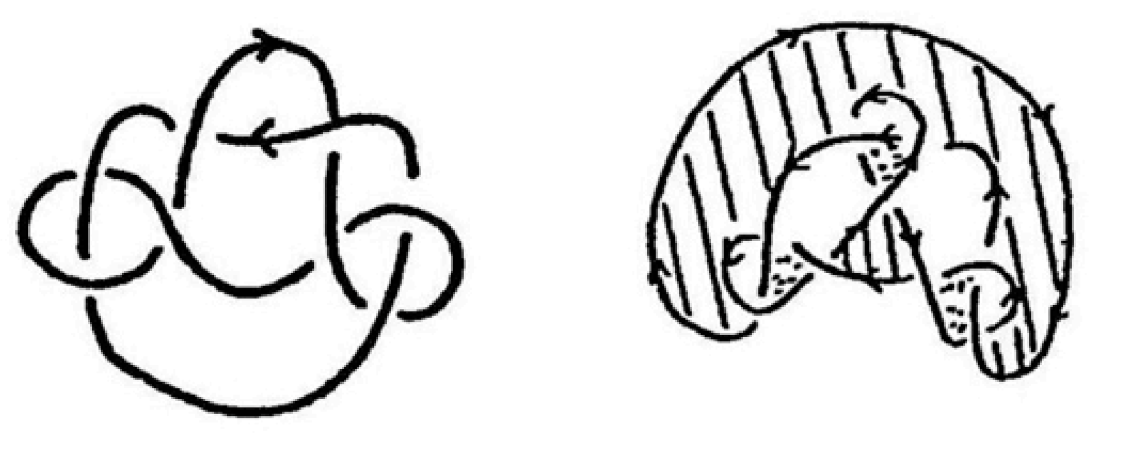
$8^2_6$①



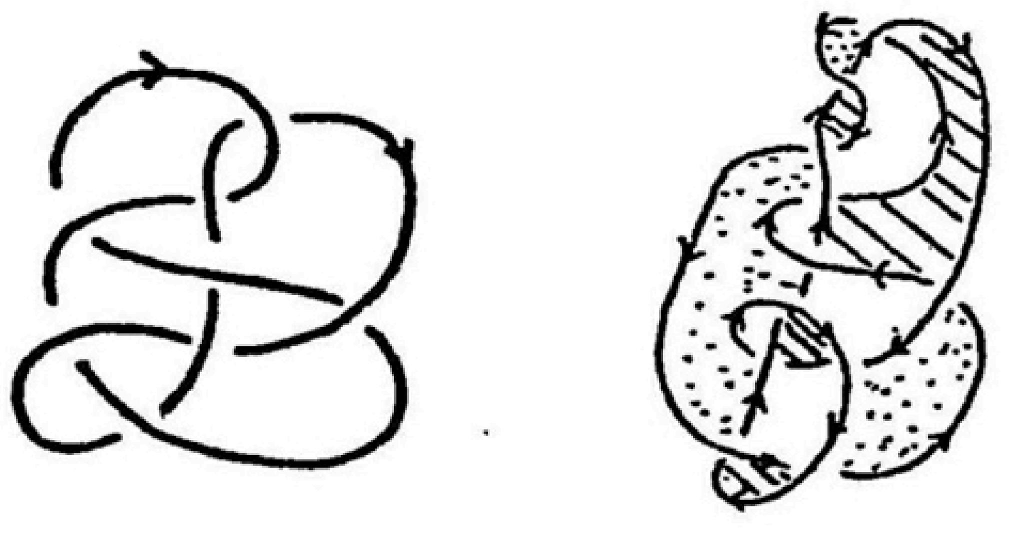
$8^2_6$②



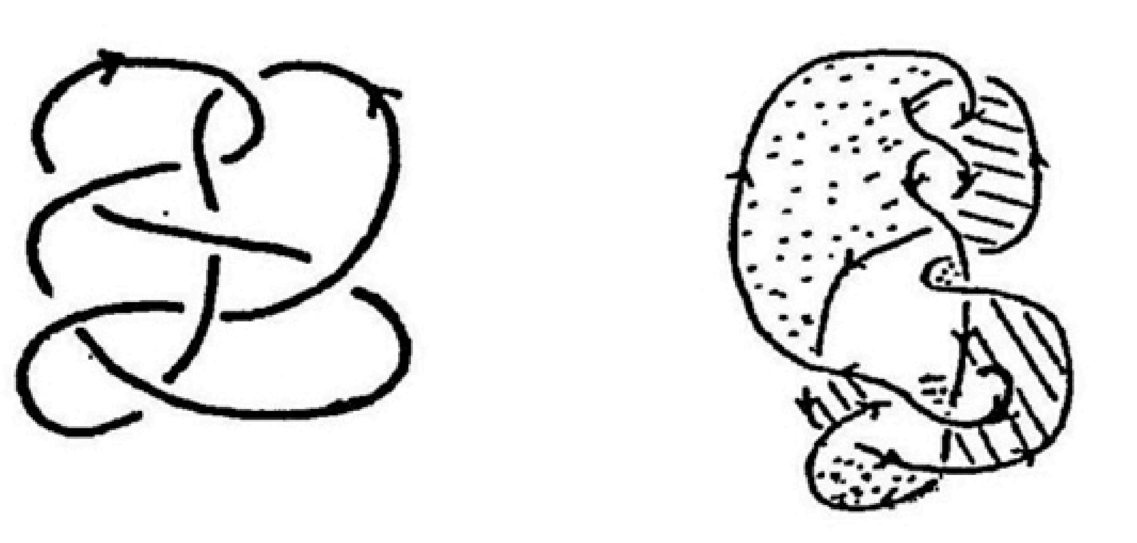
$8^2_7$①



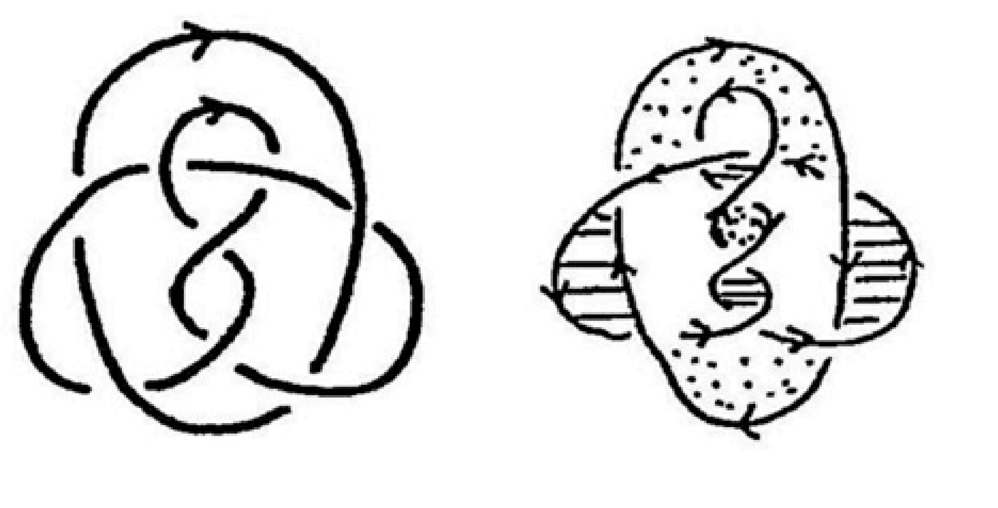
$8^2_7$②



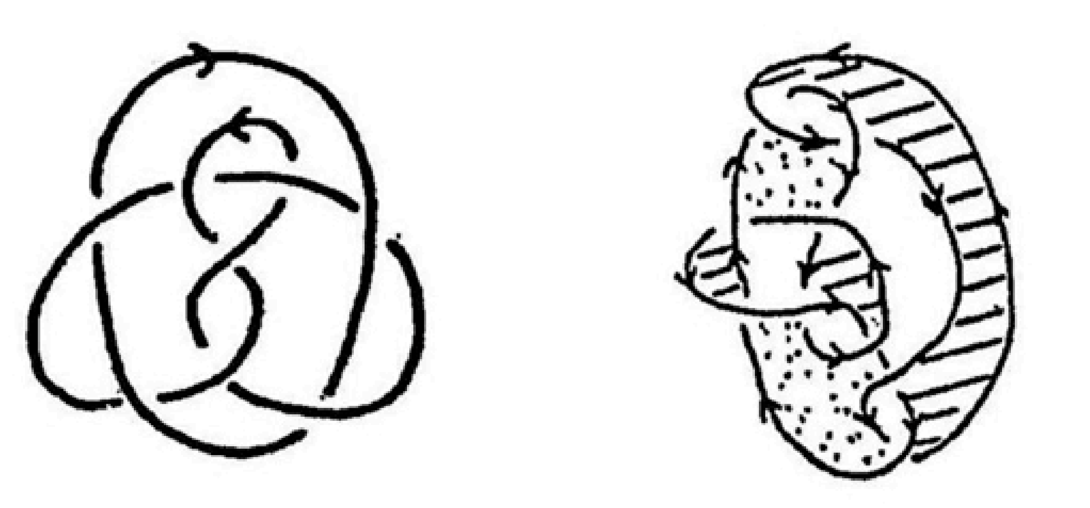
$8^2_8$①



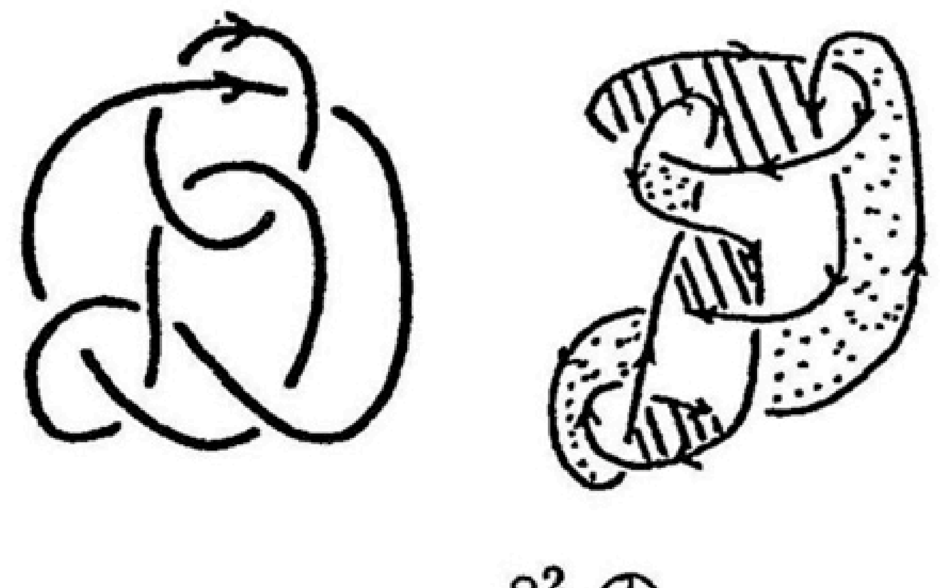
$8^2_8$②



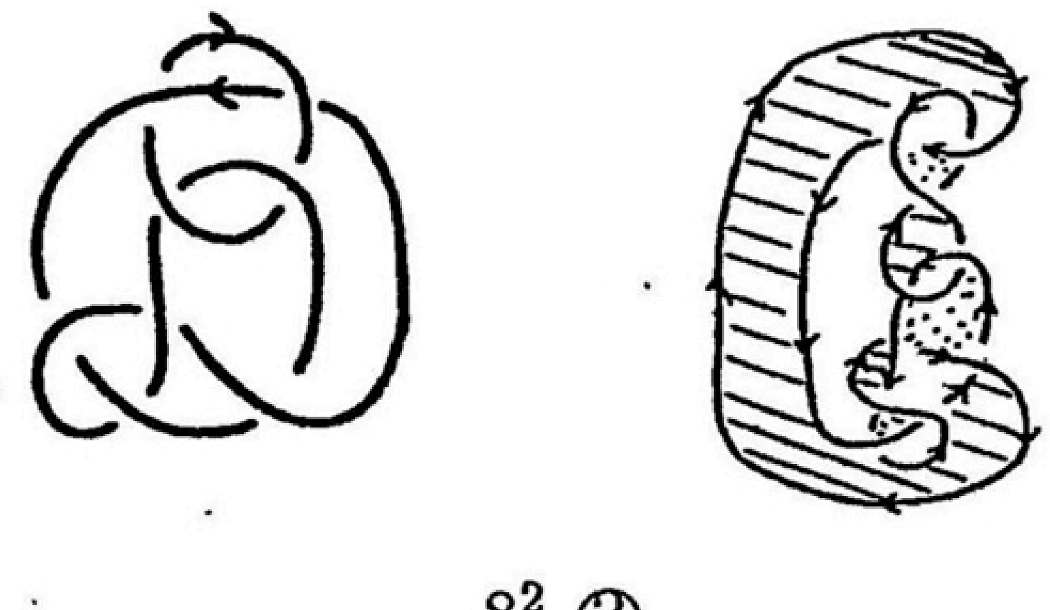
$8^2_9$①



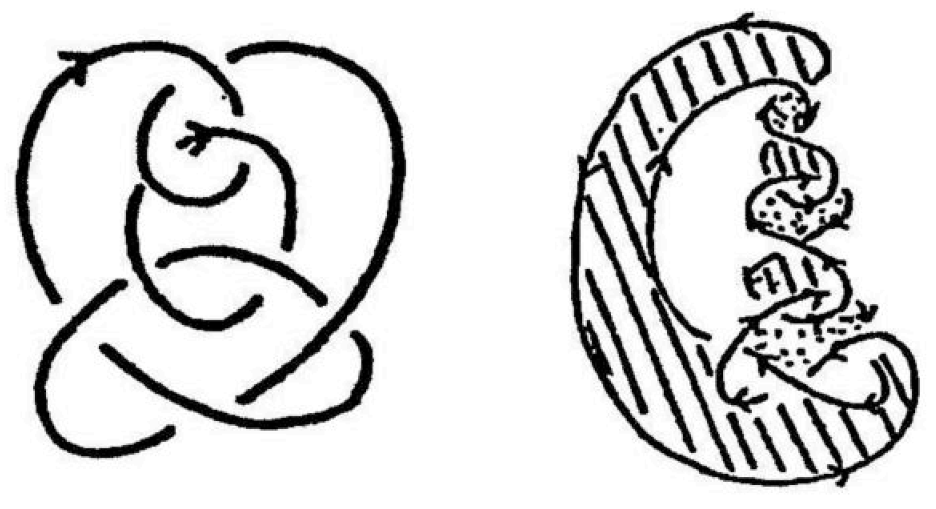
$8^2_9$②



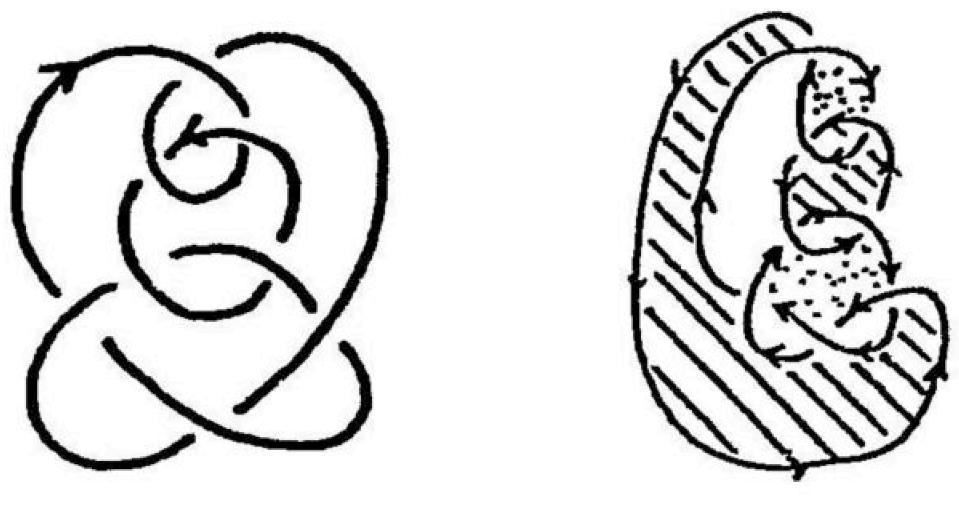
8^2_{10}①



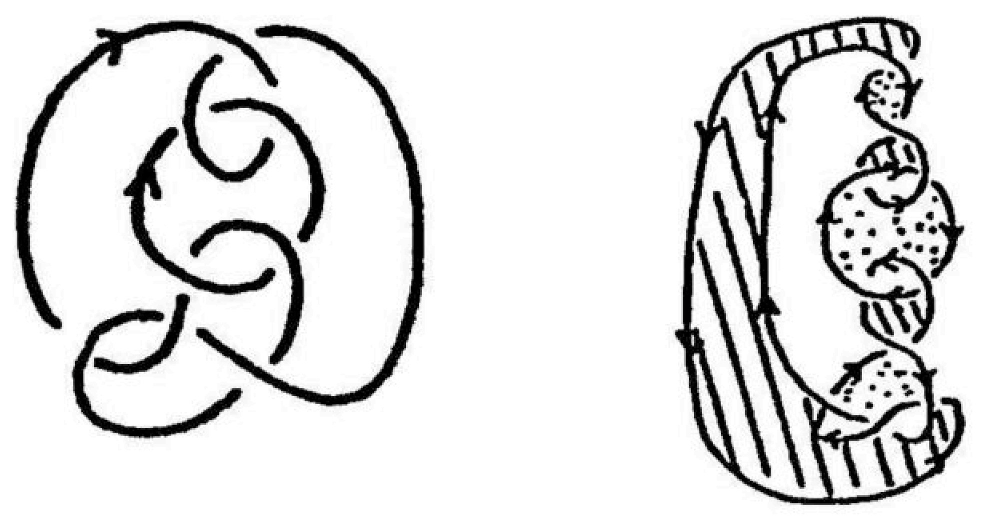
8^2_{10}②



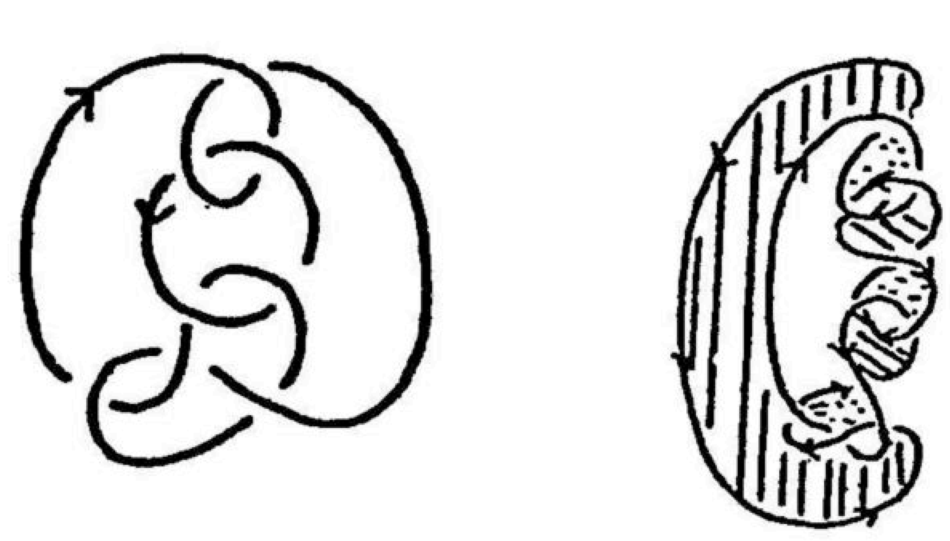
8^2_{11}①



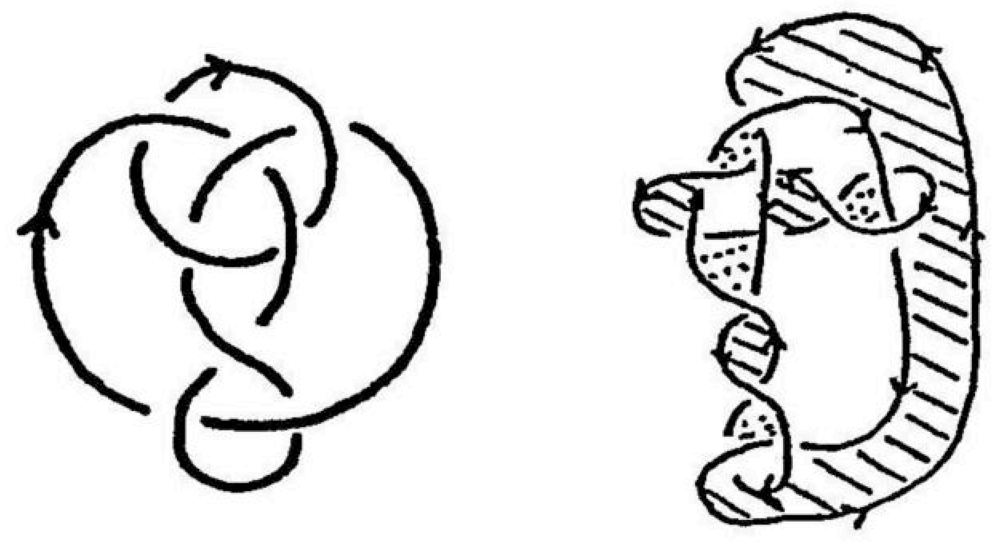
8^2_{11}②



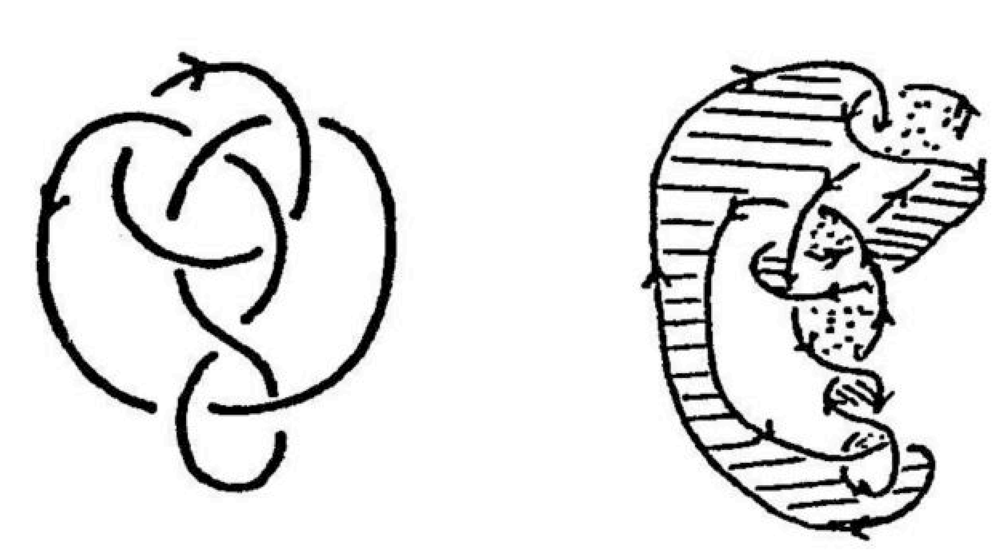
8^2_{12}①



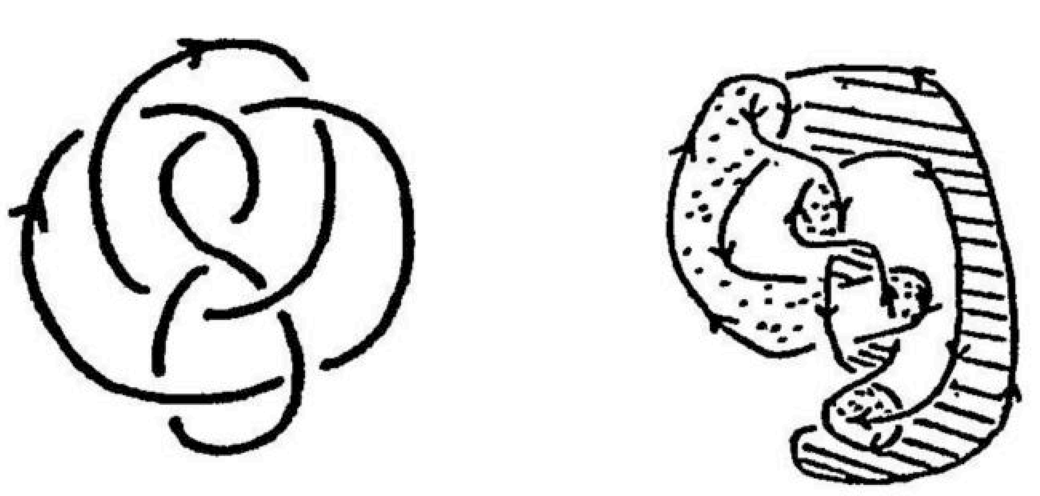
8^2_{12}②



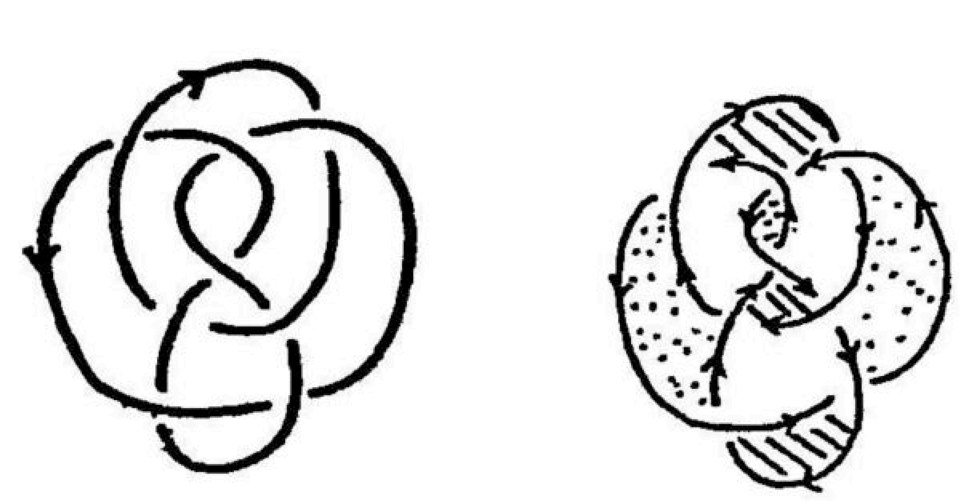
8^2_{13}①



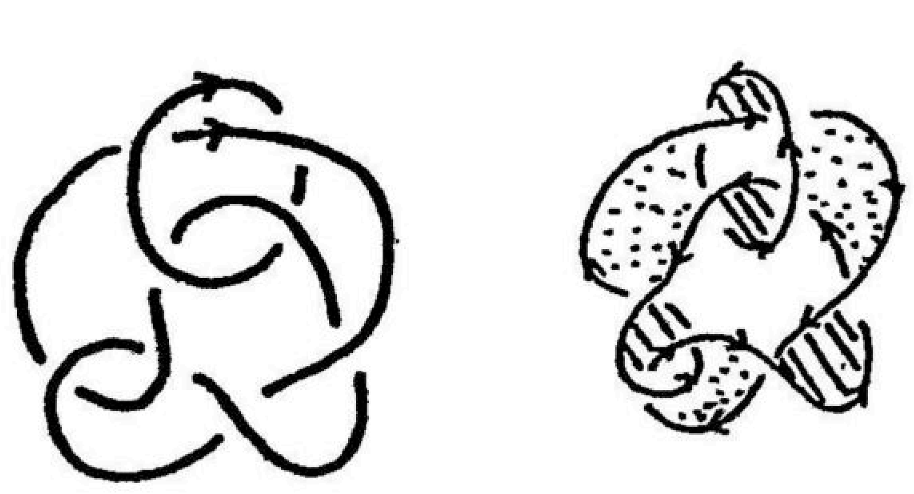
8^2_{13}②



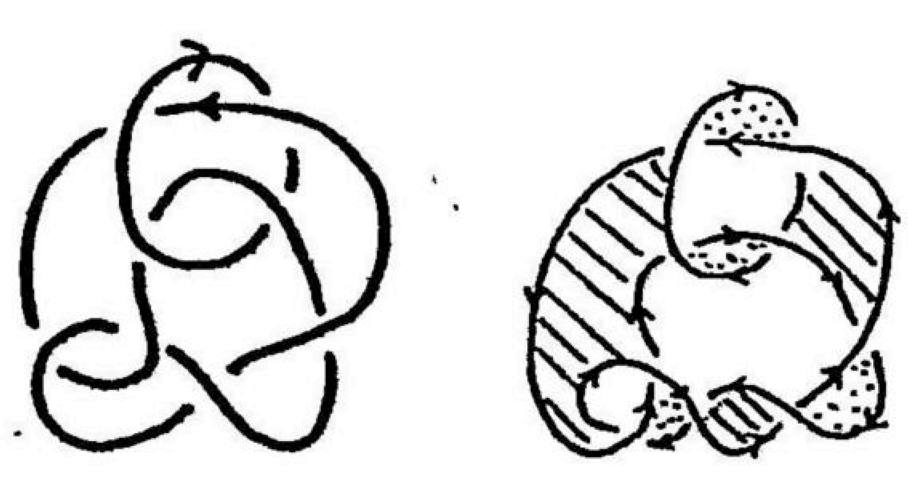
8^2_{14}①



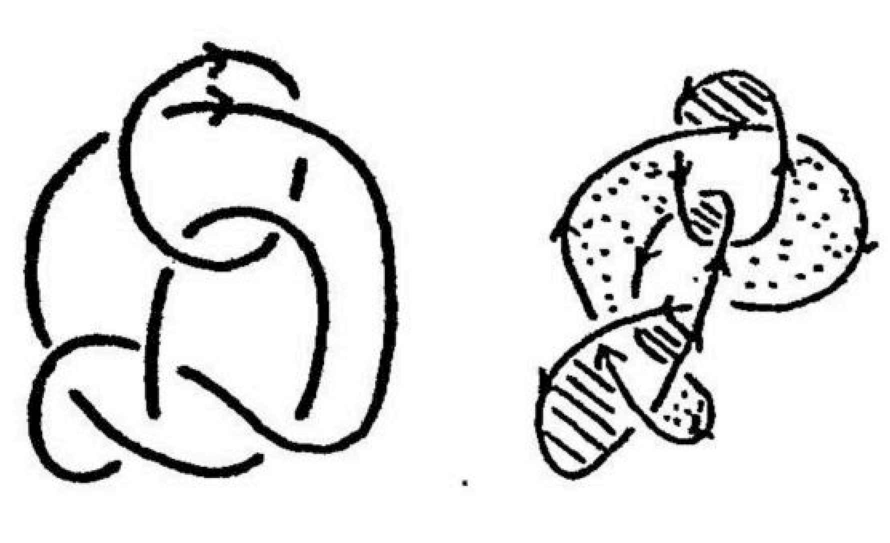
8^2_{14}②



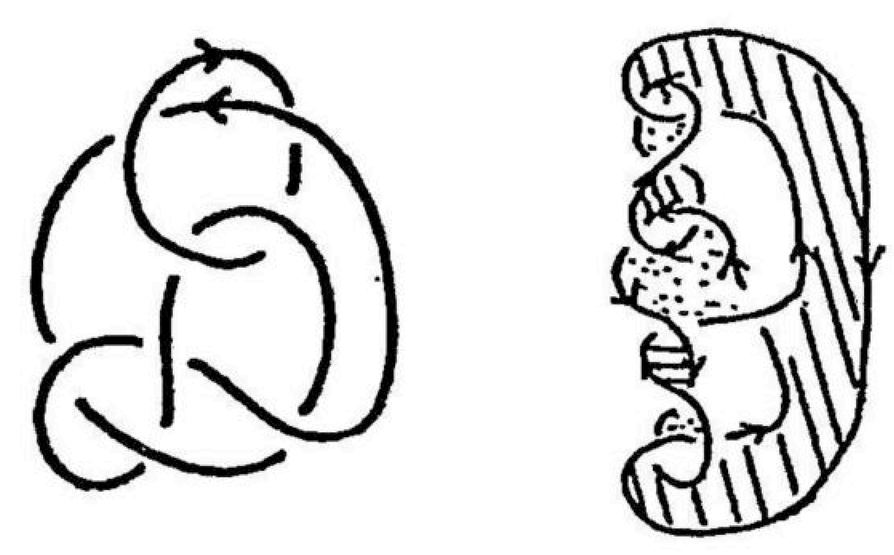
8^2_{15}①



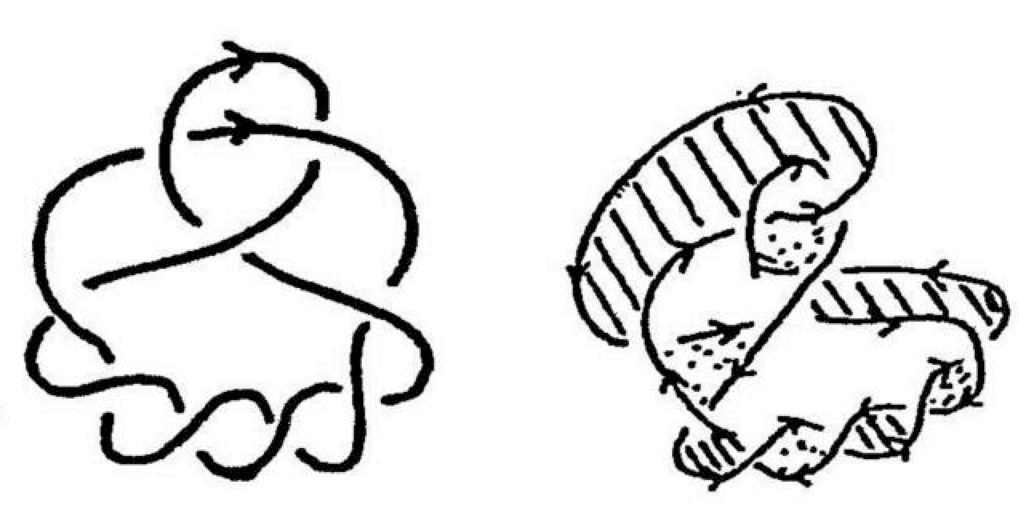
8^2_{15}②



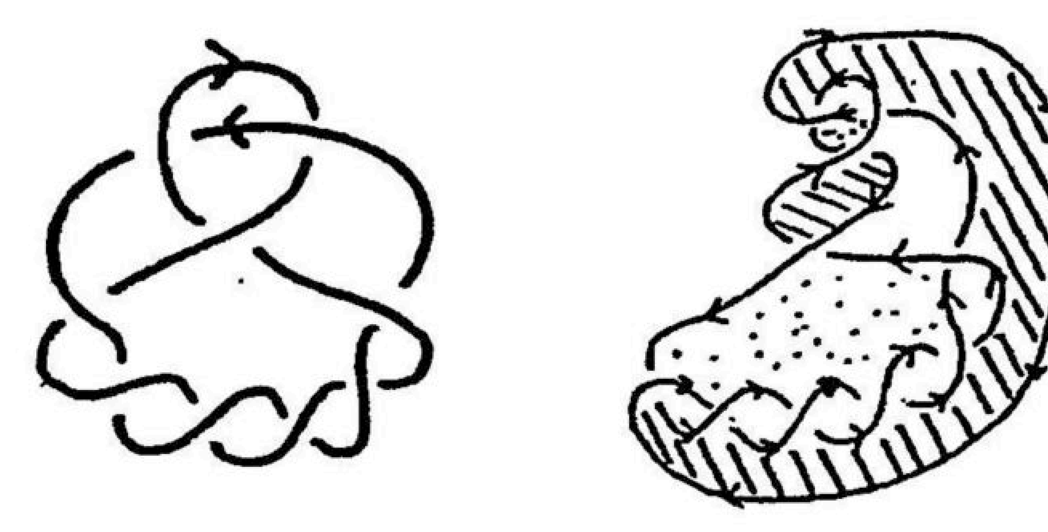
8^2_{16}①



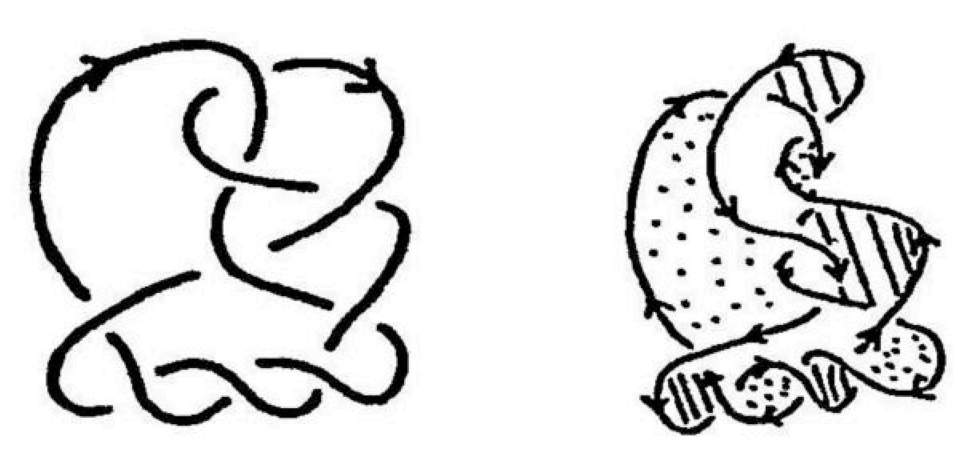
8^2_{16}②



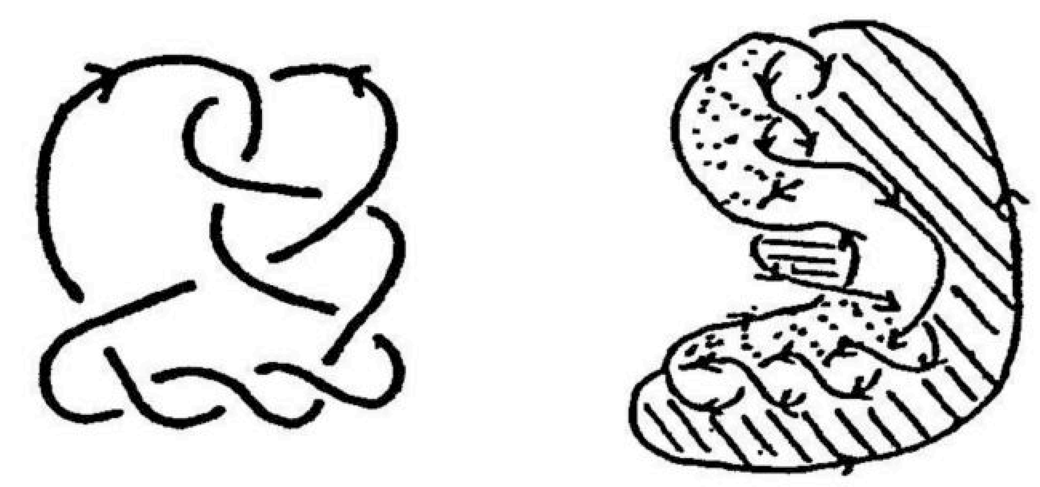
$9^2_1$①



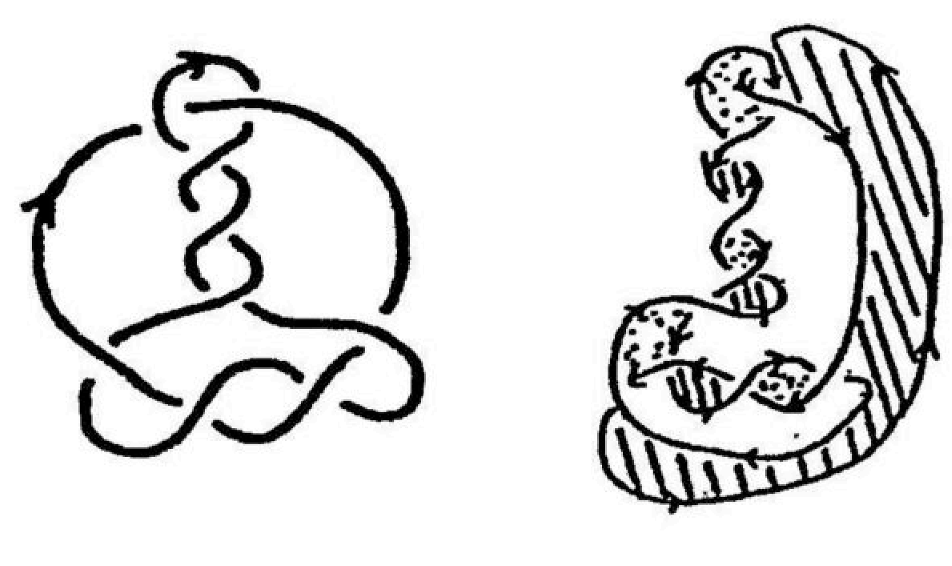
$9^2_1$②



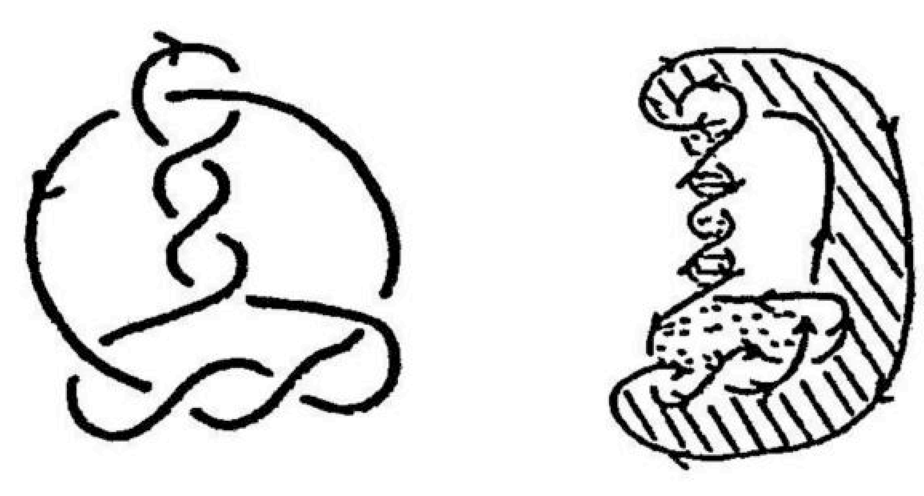
$9^2_2$①



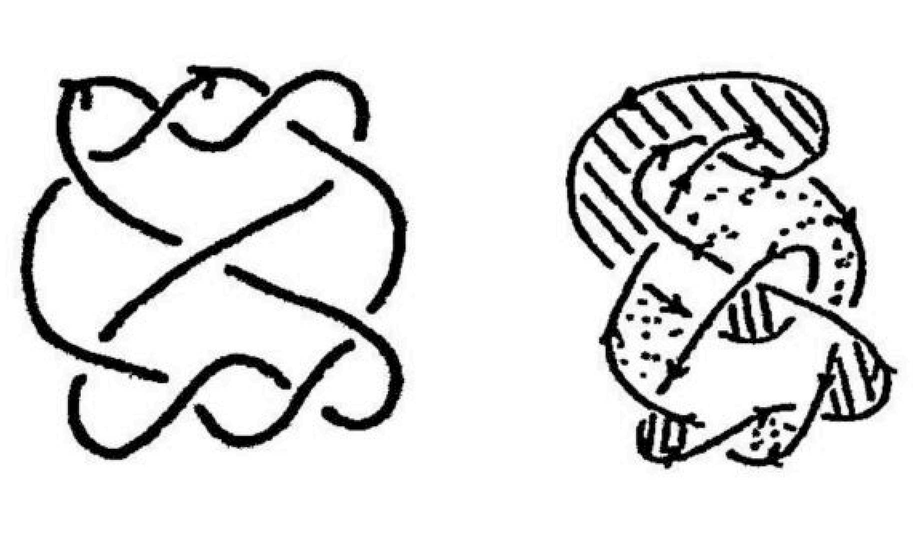
$9^2_2$②



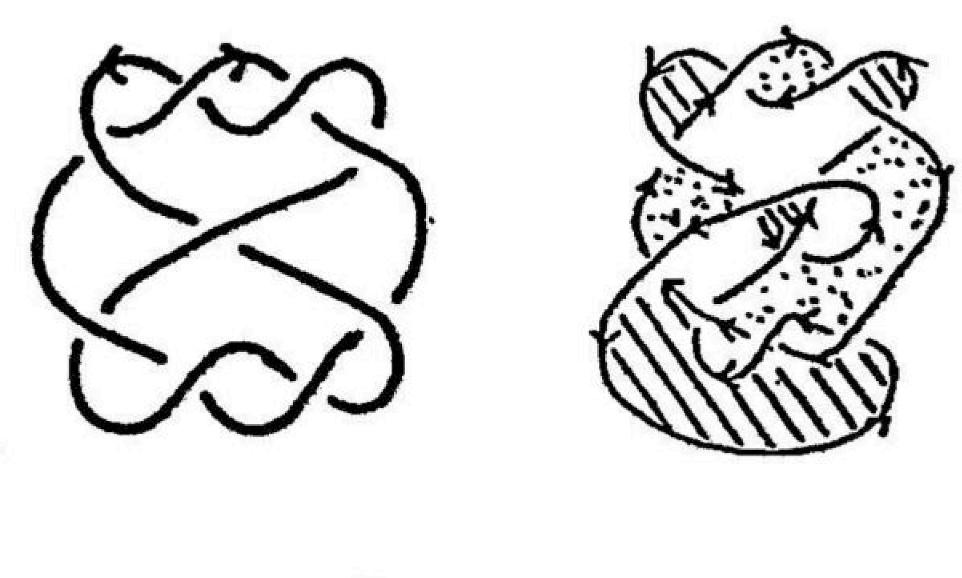
$9^2_3$①



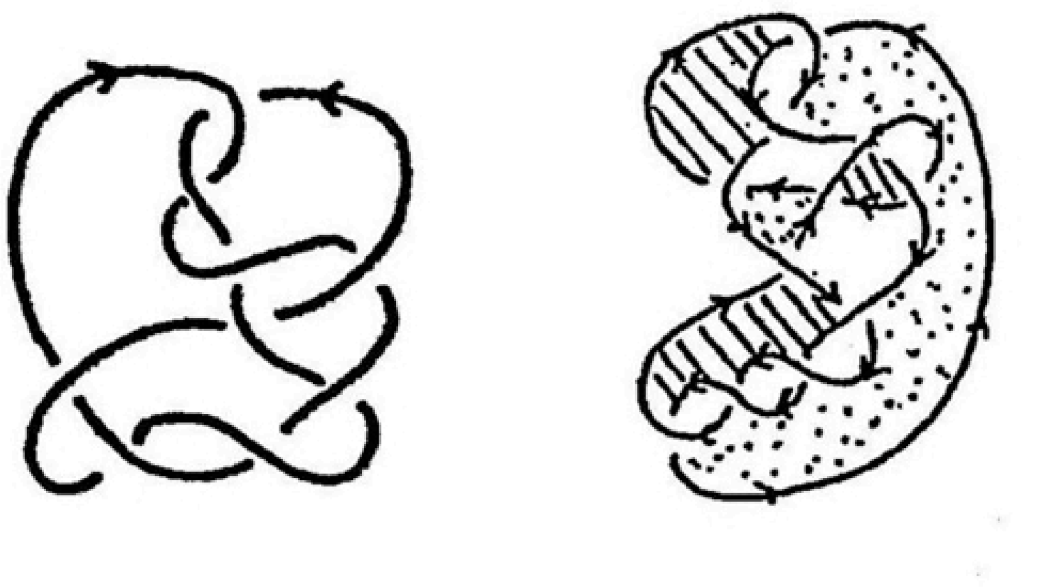
$9^2_3$②



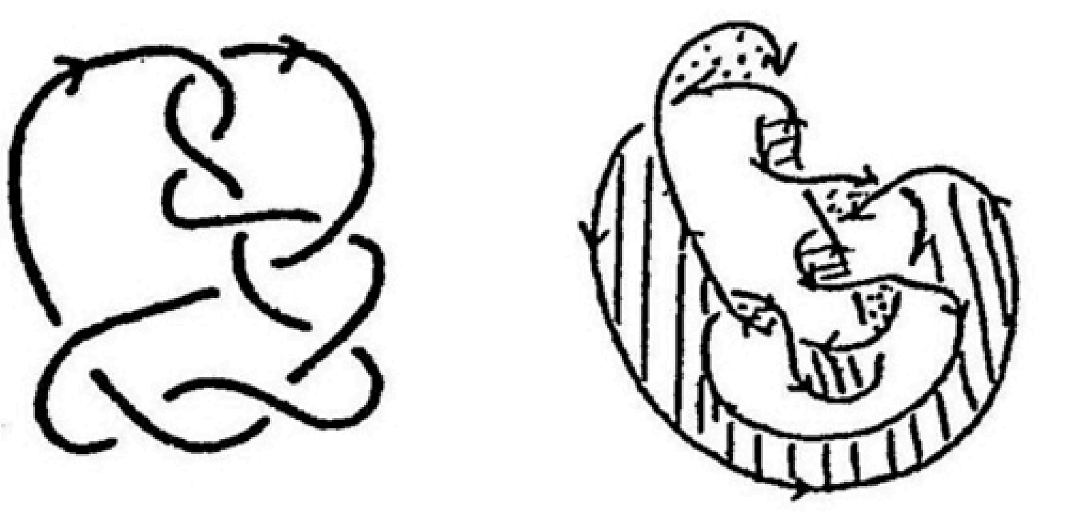
$9^2_4$①



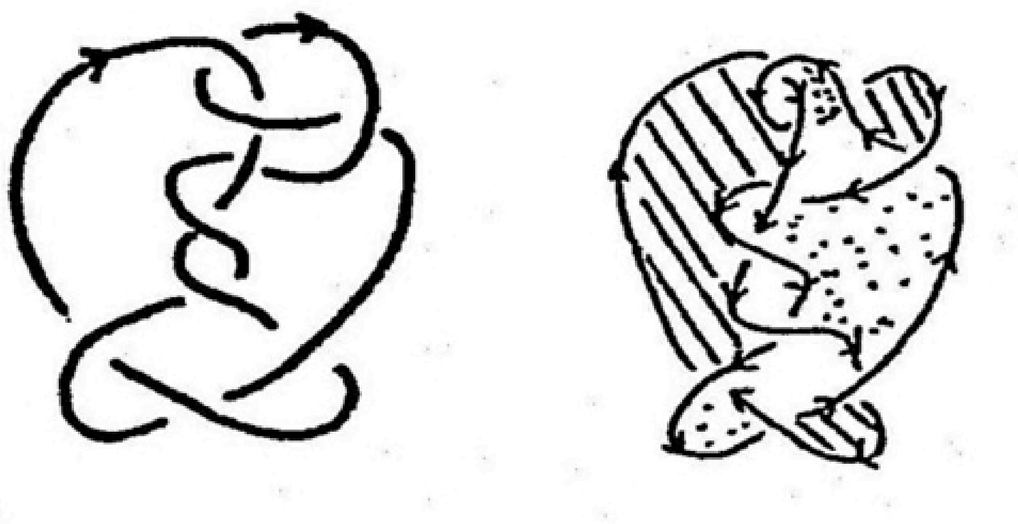
$9^2_4$②



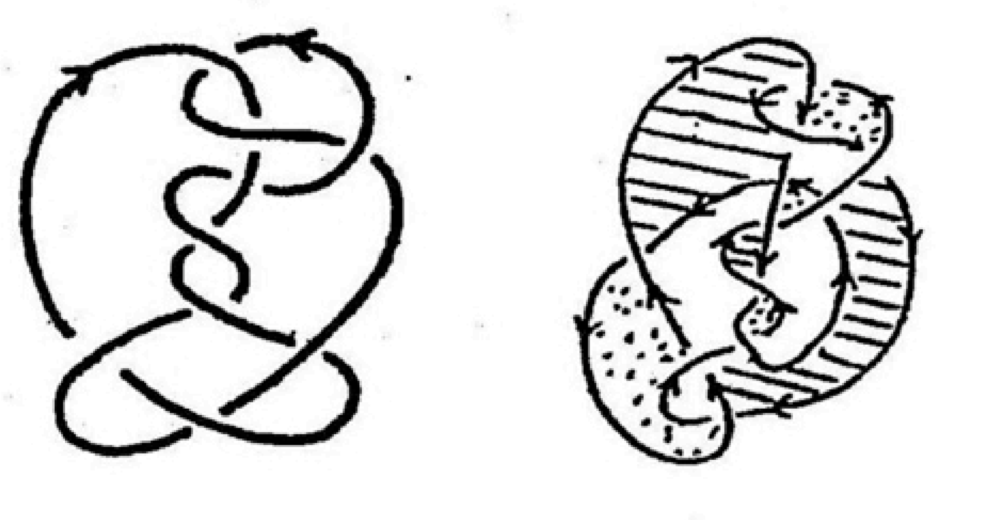
$9^2_5$①



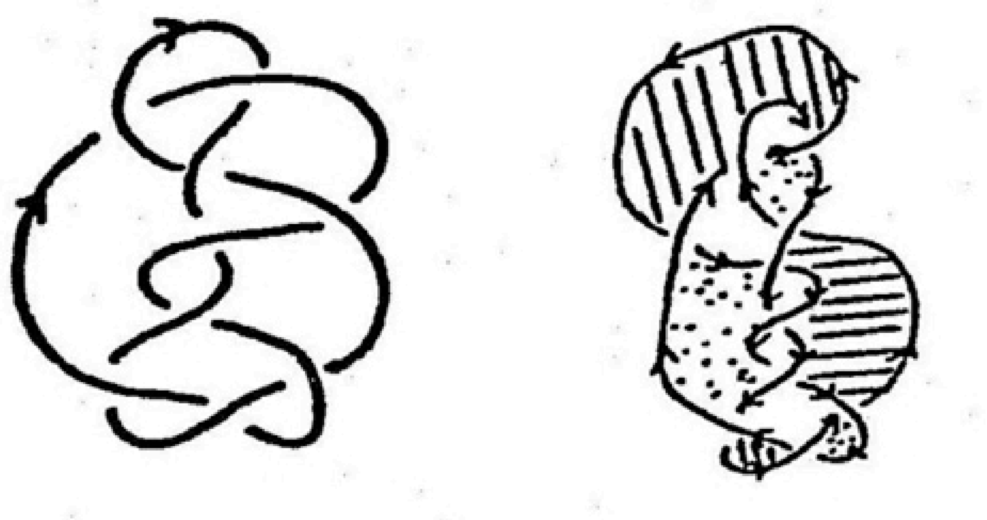
$9^2_5$②



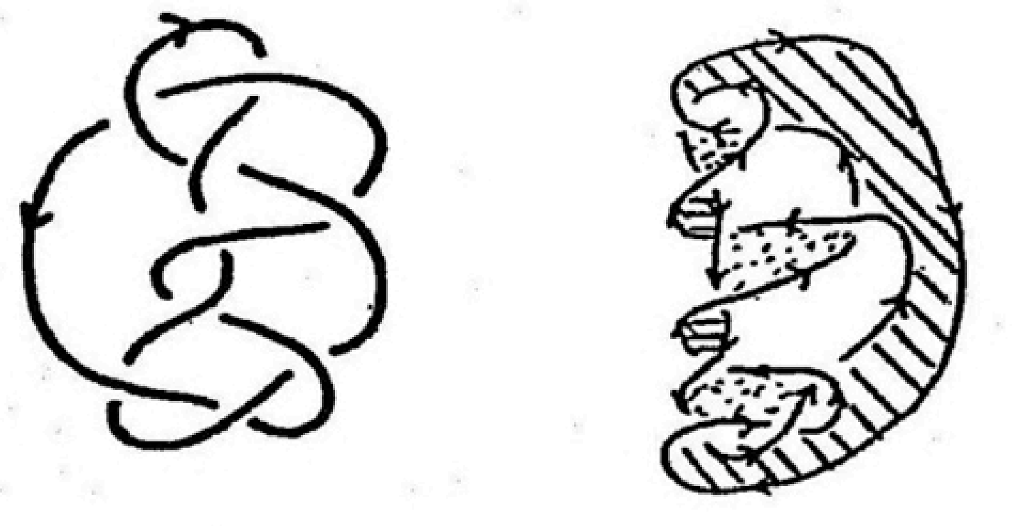
$9^2_6$①



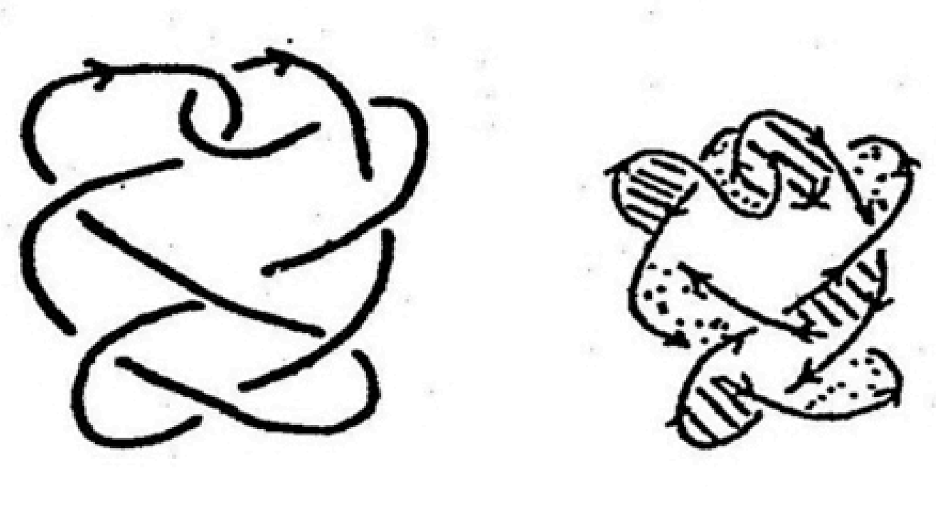
$9^2_6$②



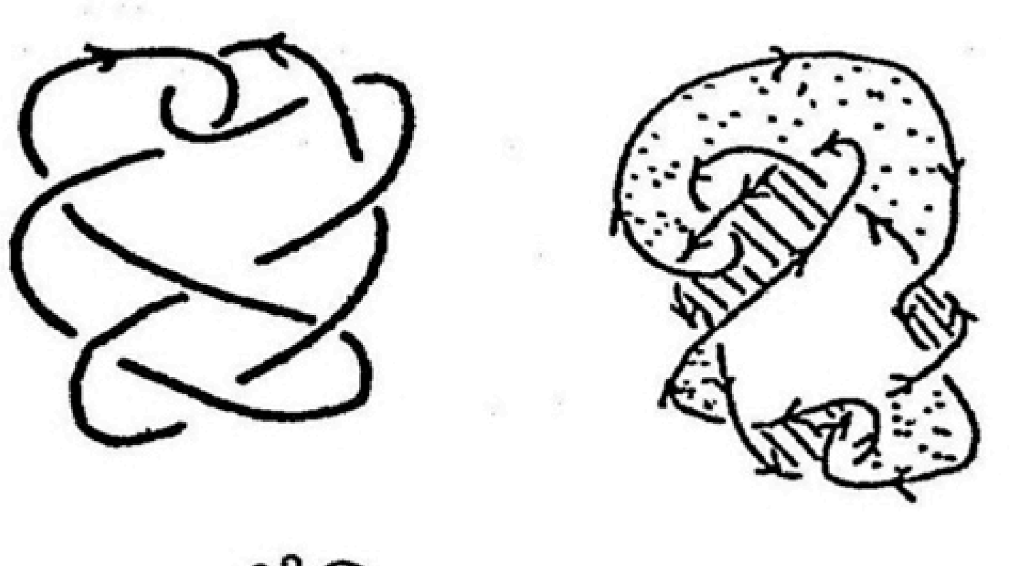
$9^2_7$①



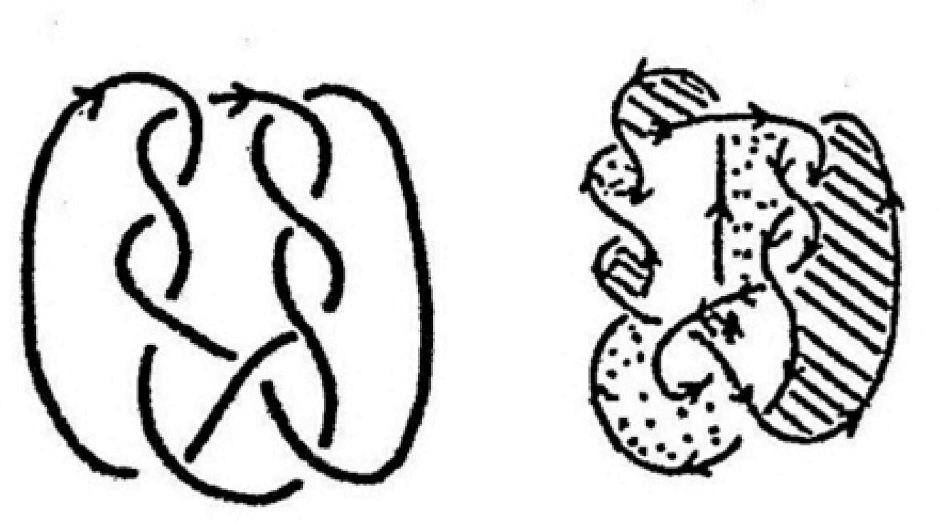
$9^2_7$②



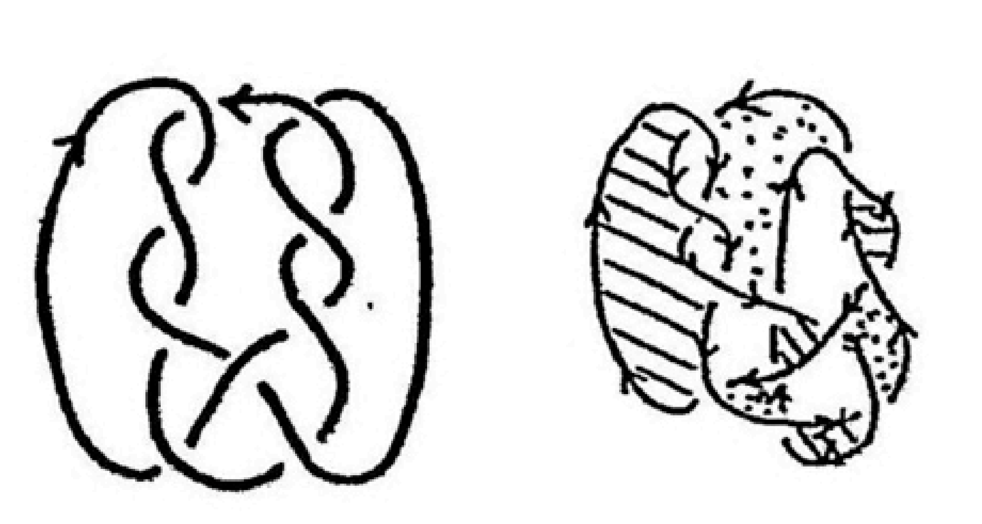
$9^2_8$①



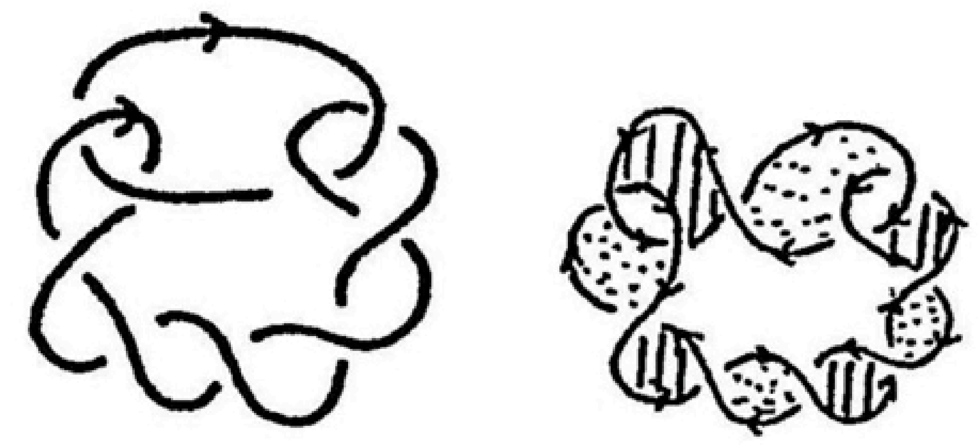
$9^2_8$②



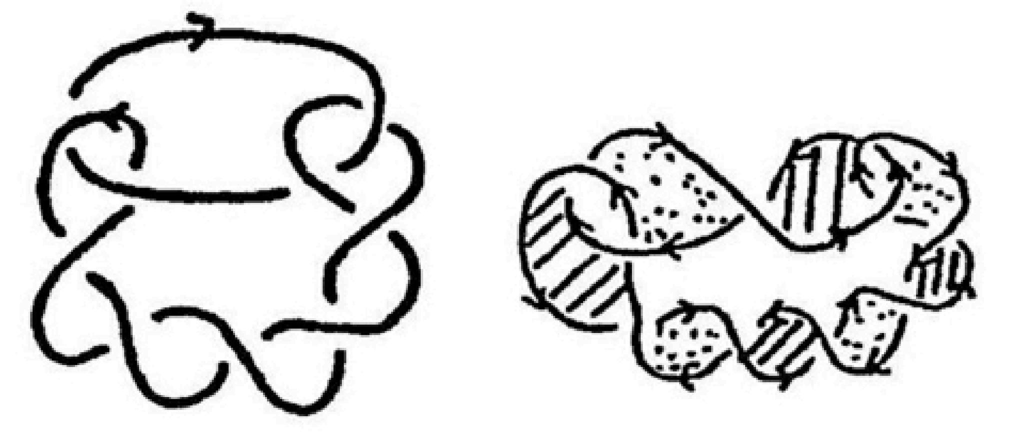
$9^2_9$①



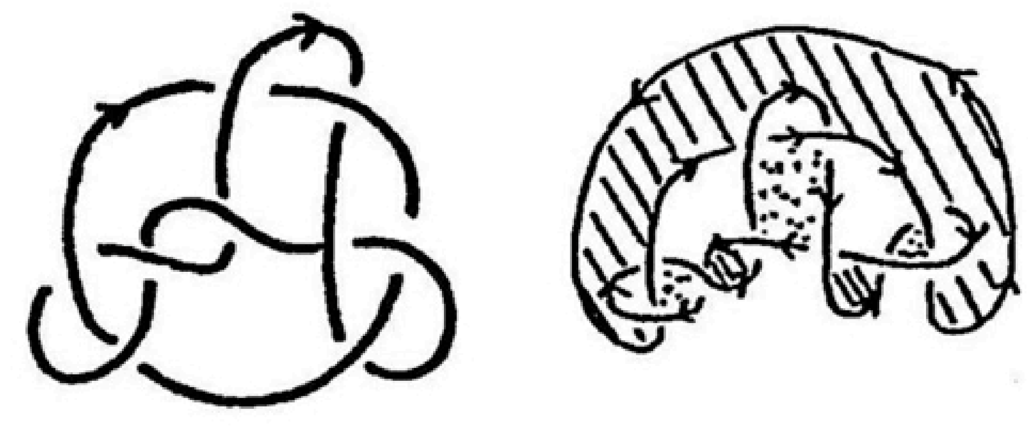
$9^2_9$②



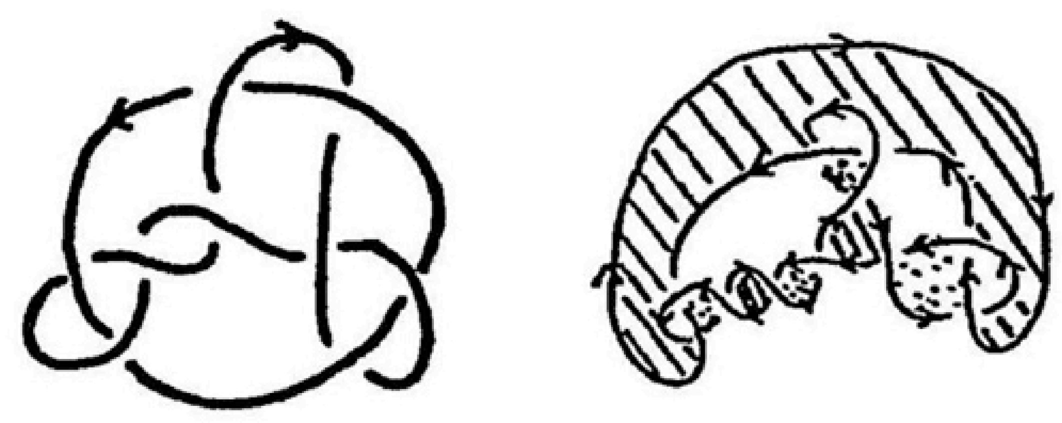
9^2_{10}①



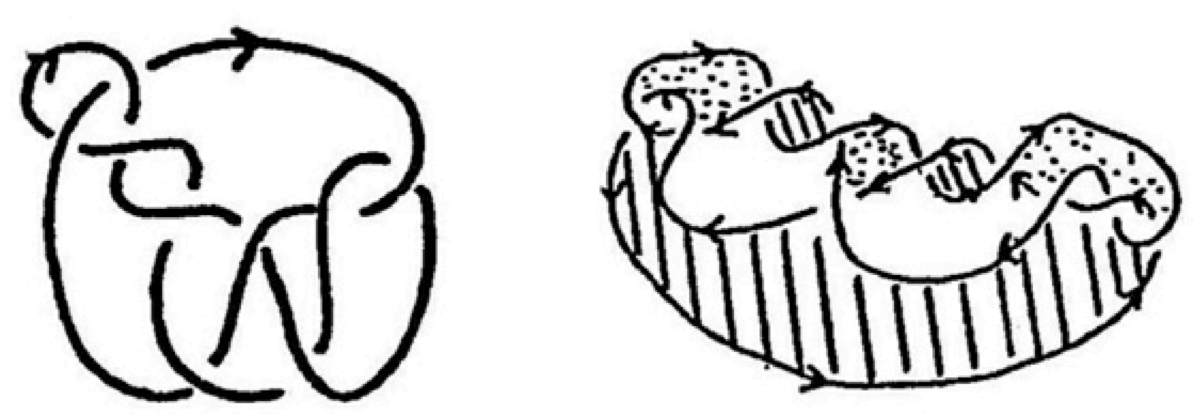
9^2_{10}②



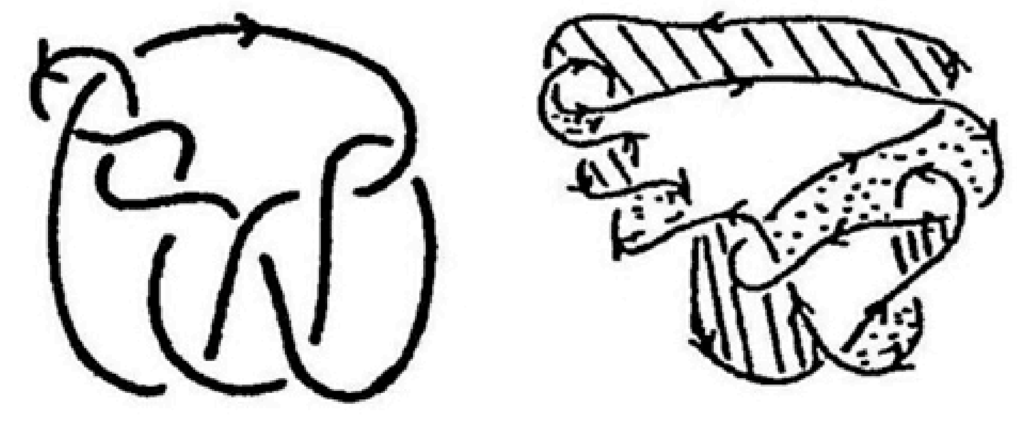
9^2_{11}①



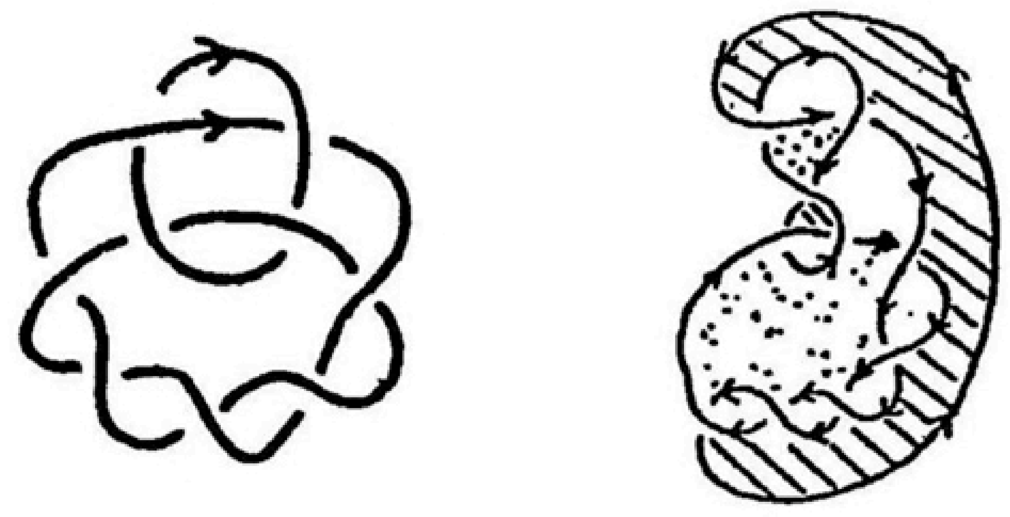
9^2_{11}②



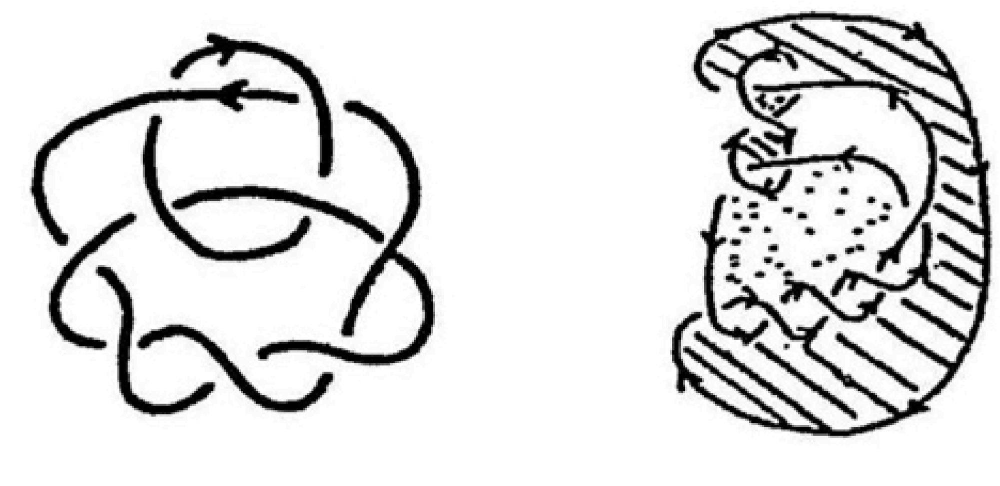
9^2_{12}①



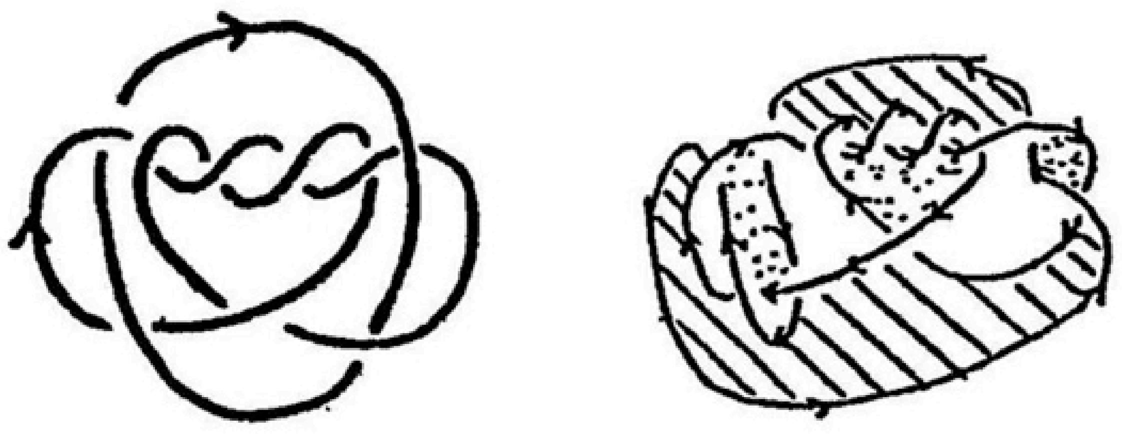
9^2_{12}②



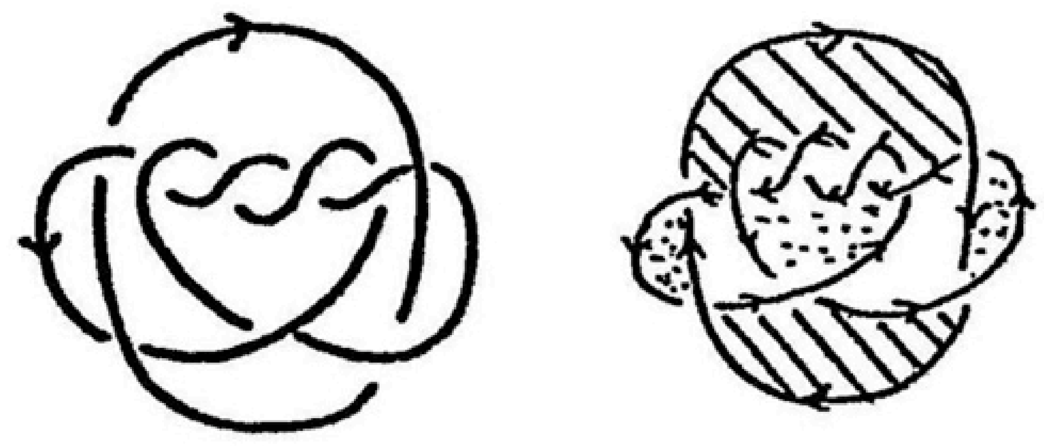
9^2_{13}①



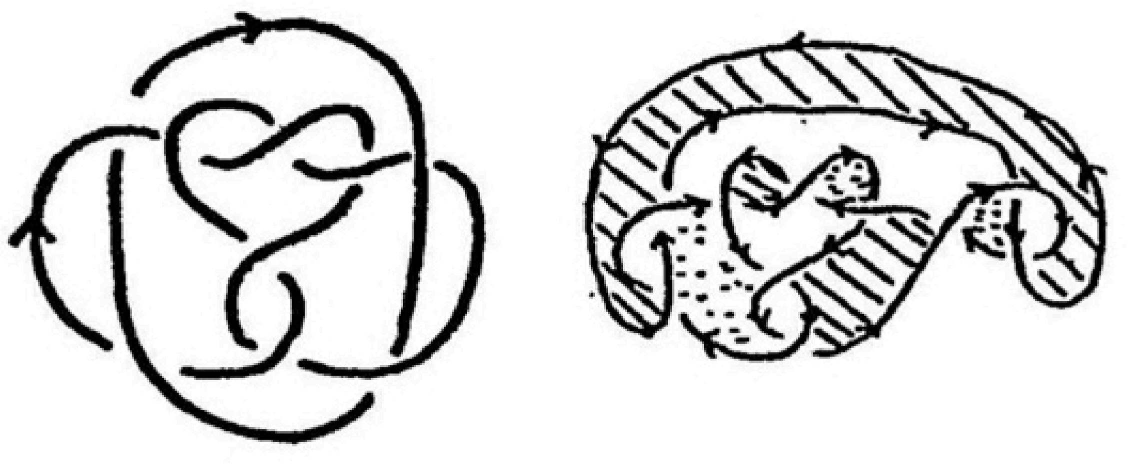
9^2_{13}②



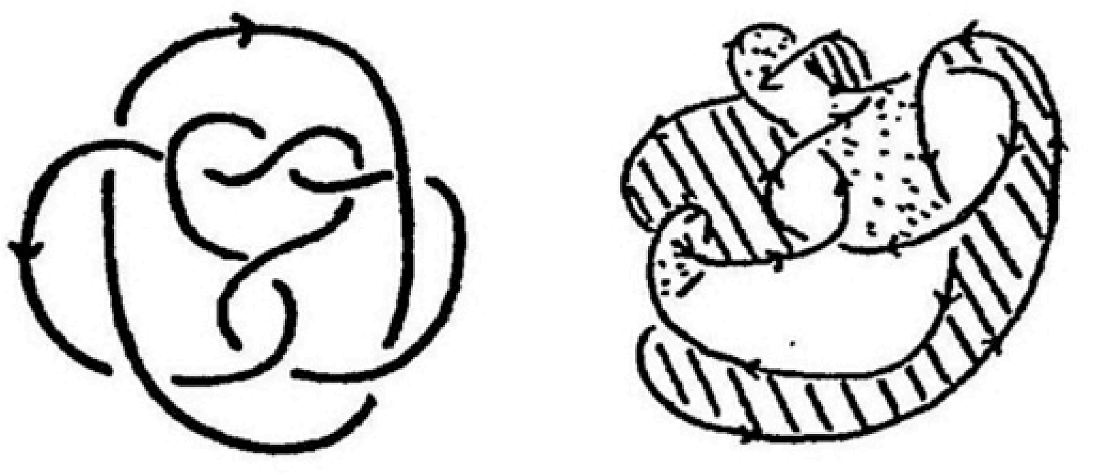
9^2_{14}①



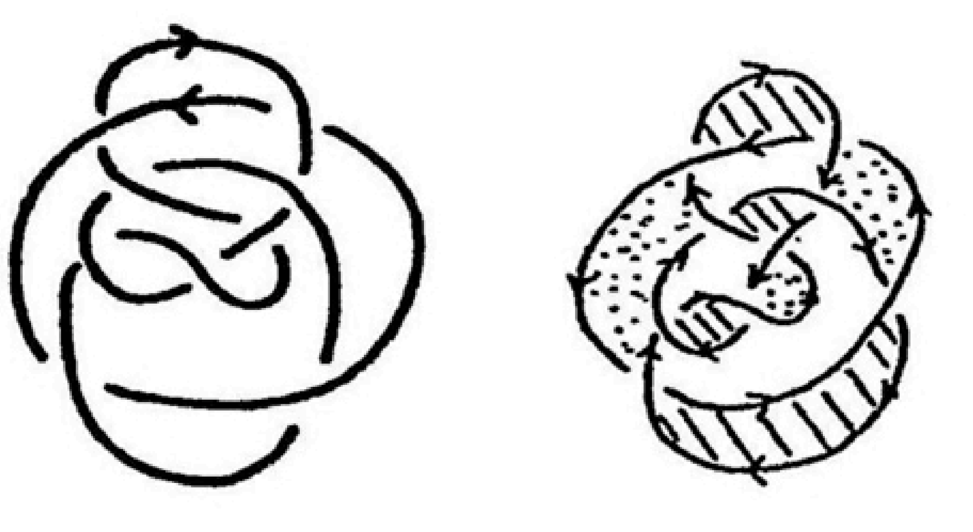
9^2_{14}②



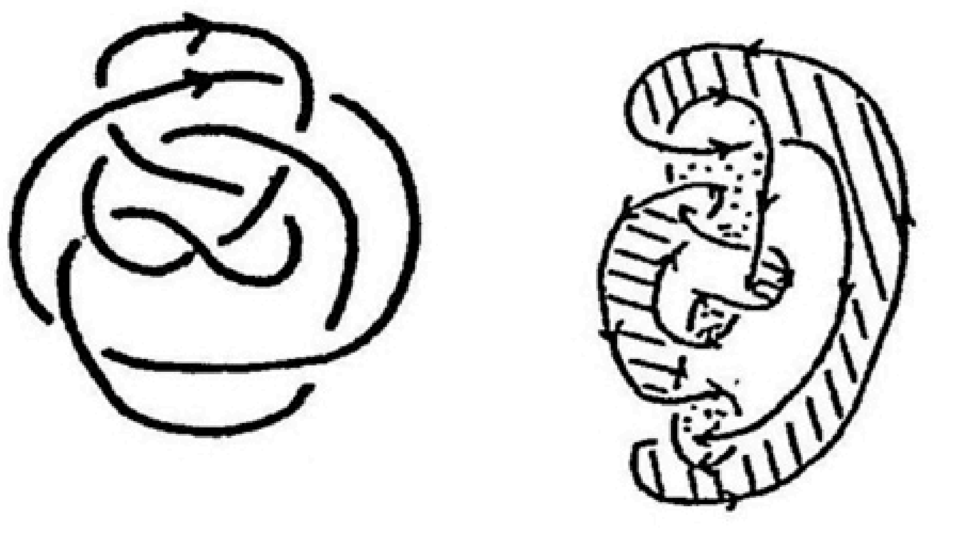
9^2_{15}①



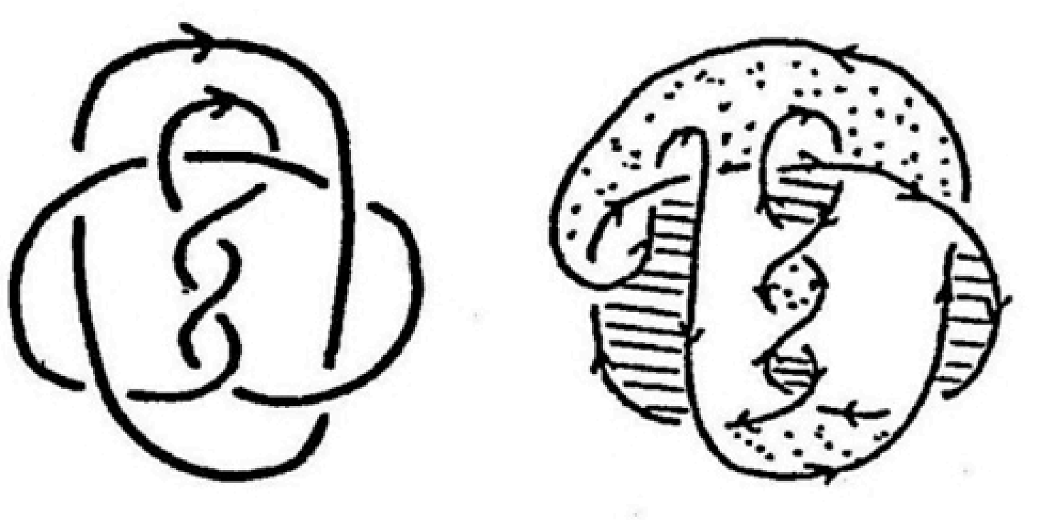
9^2_{15}②



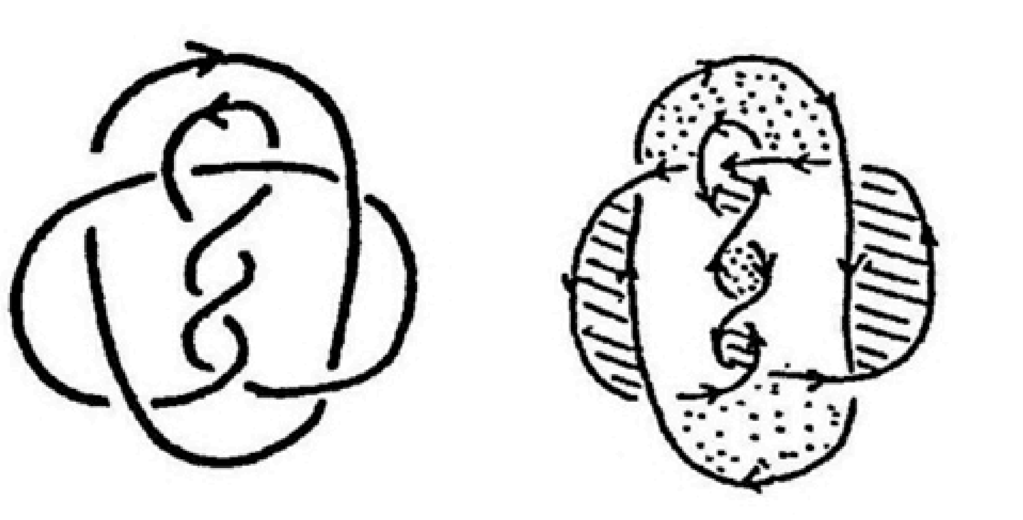
9^2_{16}①



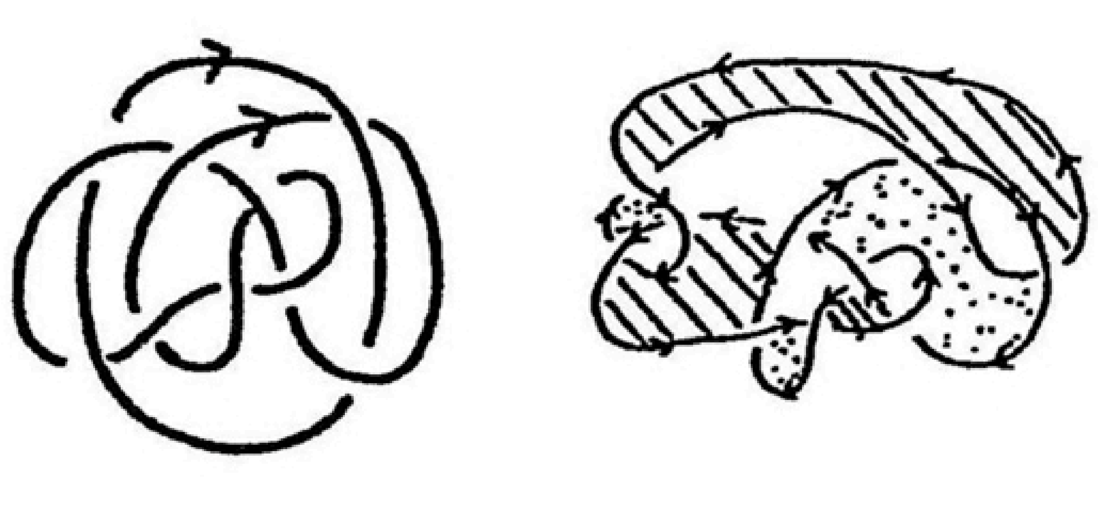
9^2_{16}②



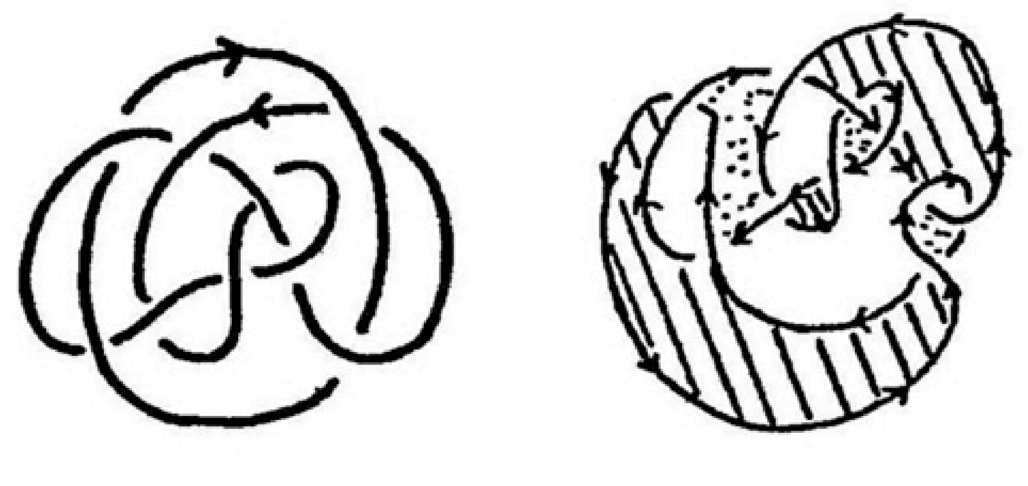
9^2_{17}①



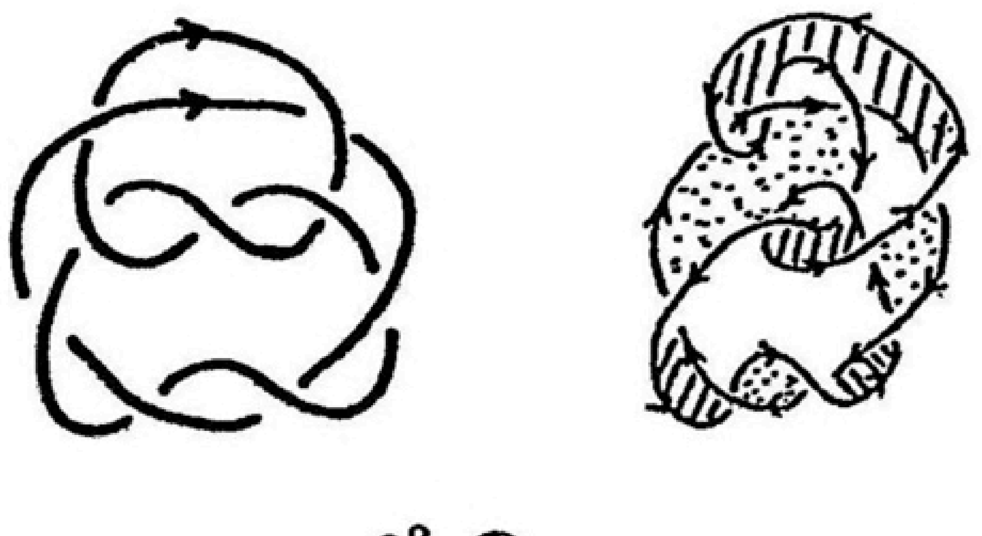
9^2_{17}②



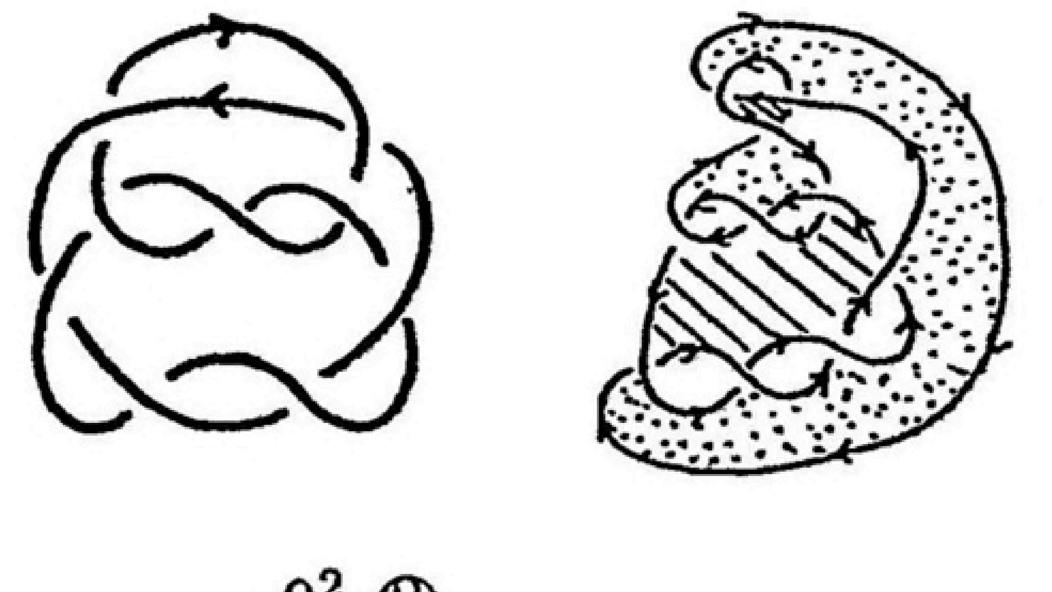
9^2_{18}①



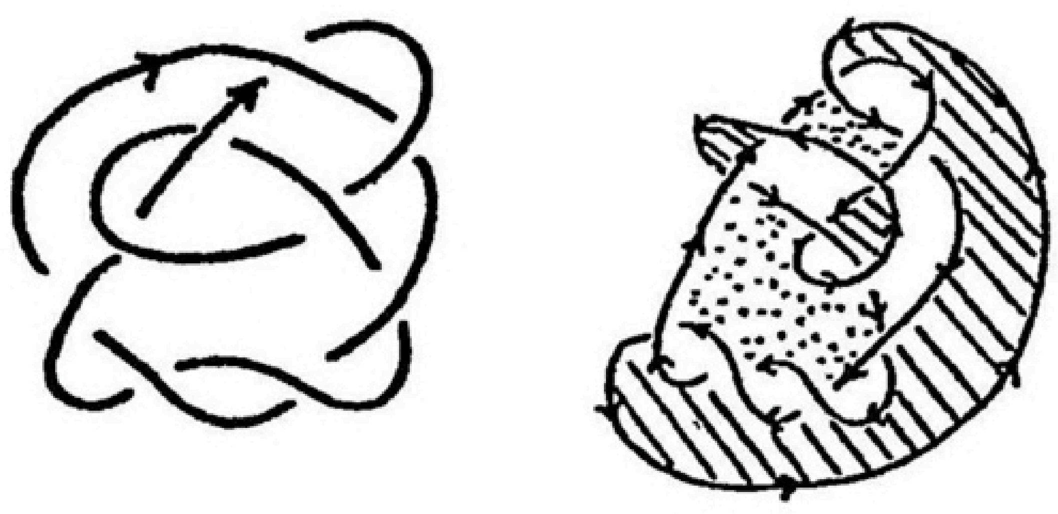
9^2_{18}②



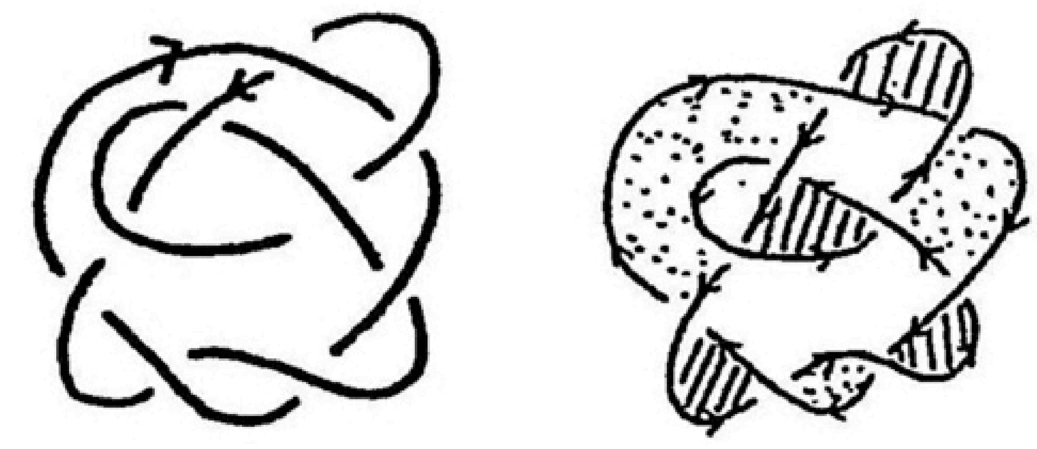
9^2_{19}①



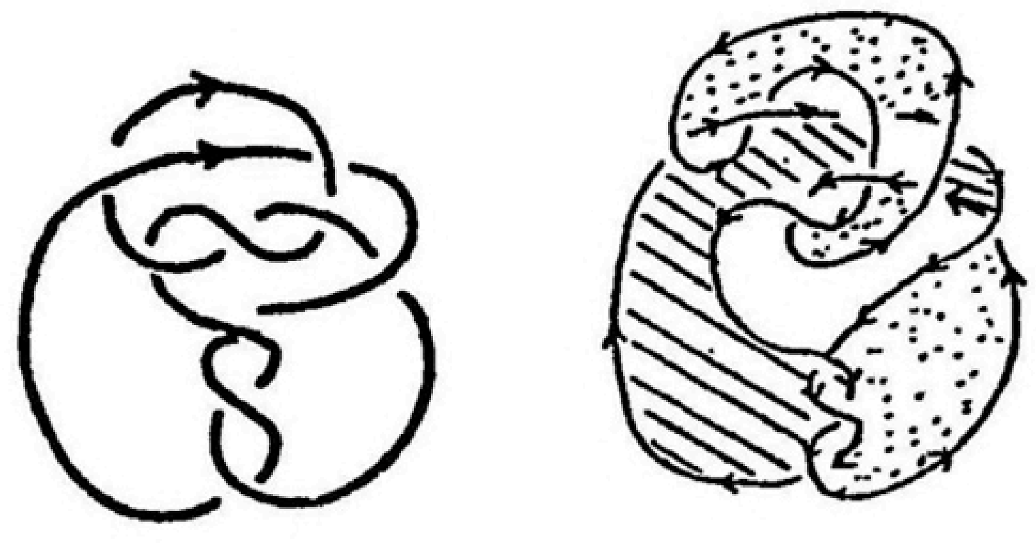
9^2_{19}②



9^2_{20}①

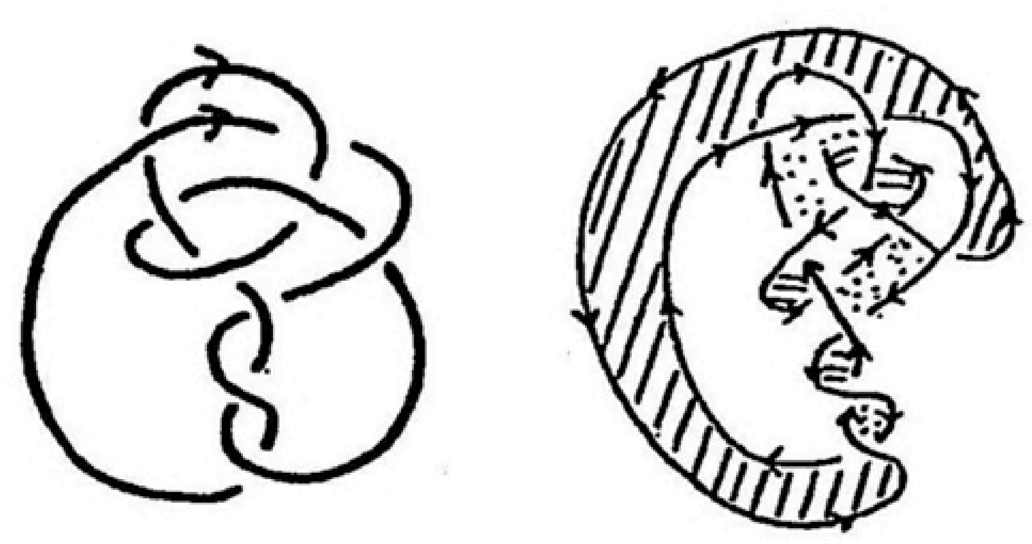


9^2_{20}②

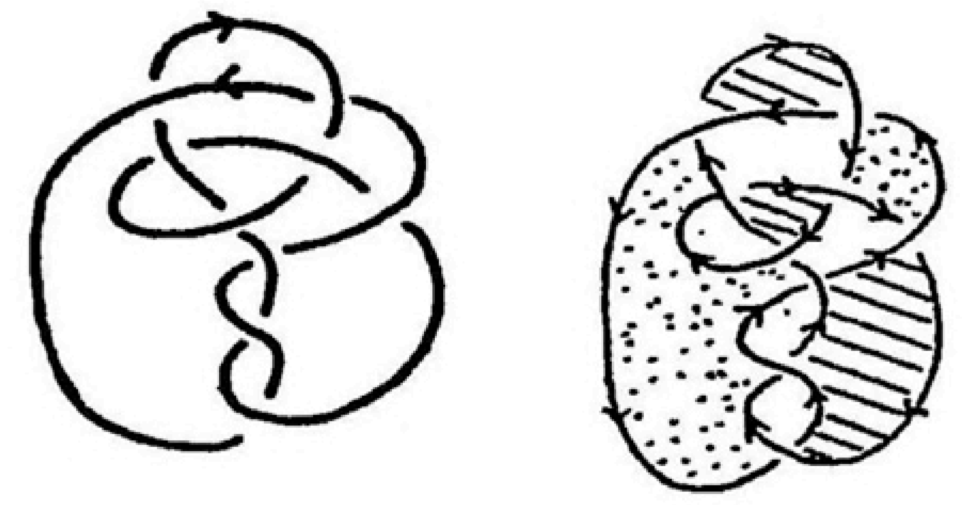


9^2_{21}①

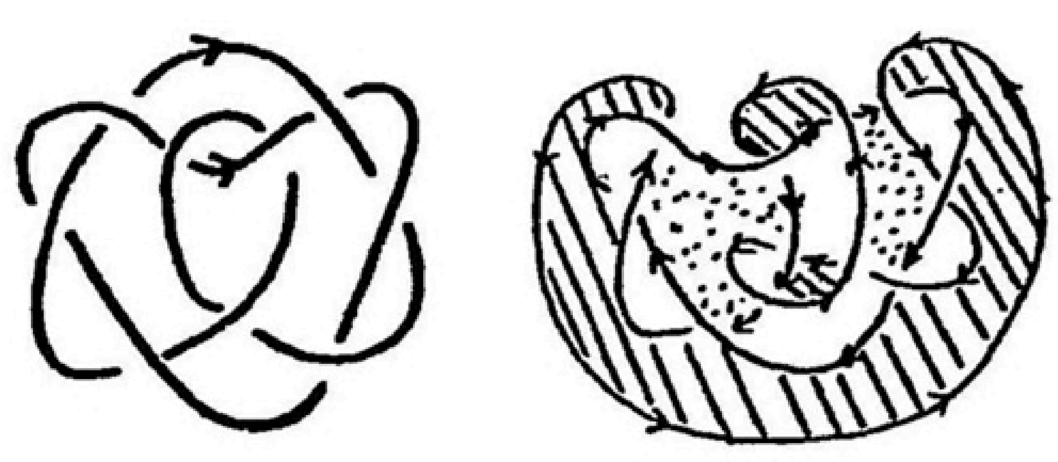
9^2_{21}②



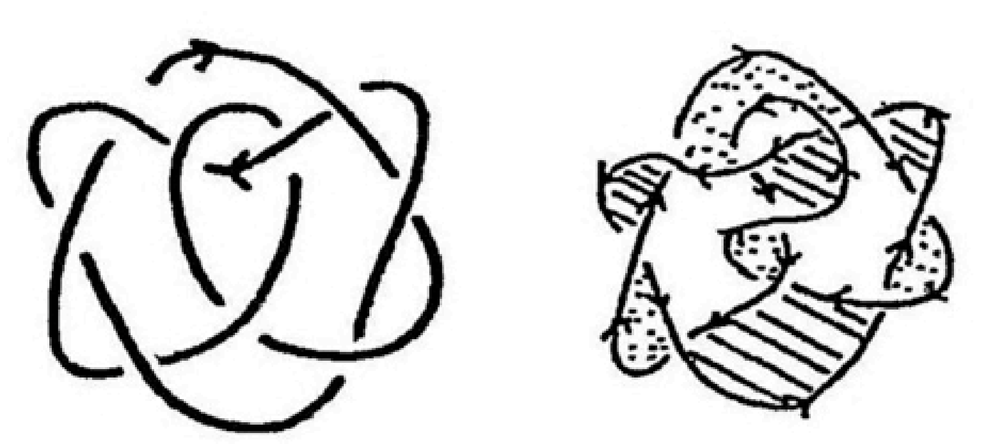
9^2_{22}①



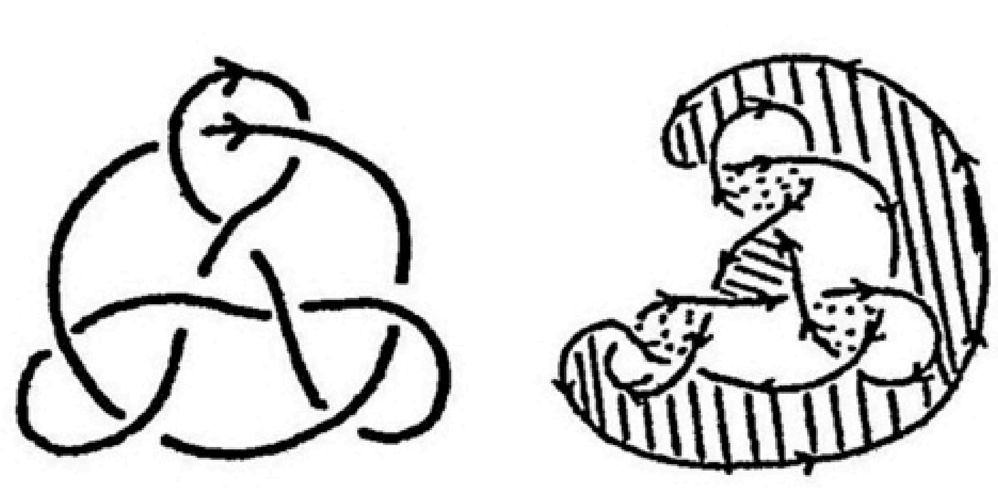
9^2_{22}②



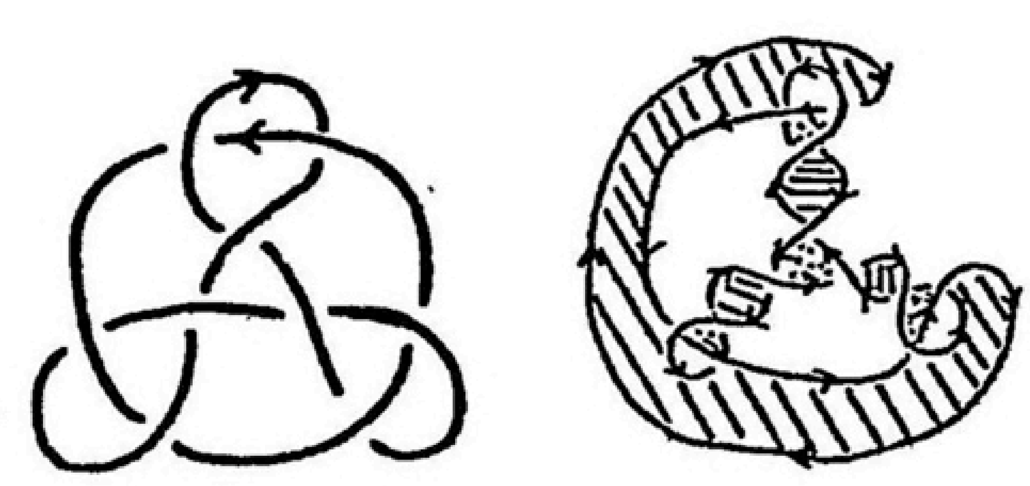
9^2_{23}①



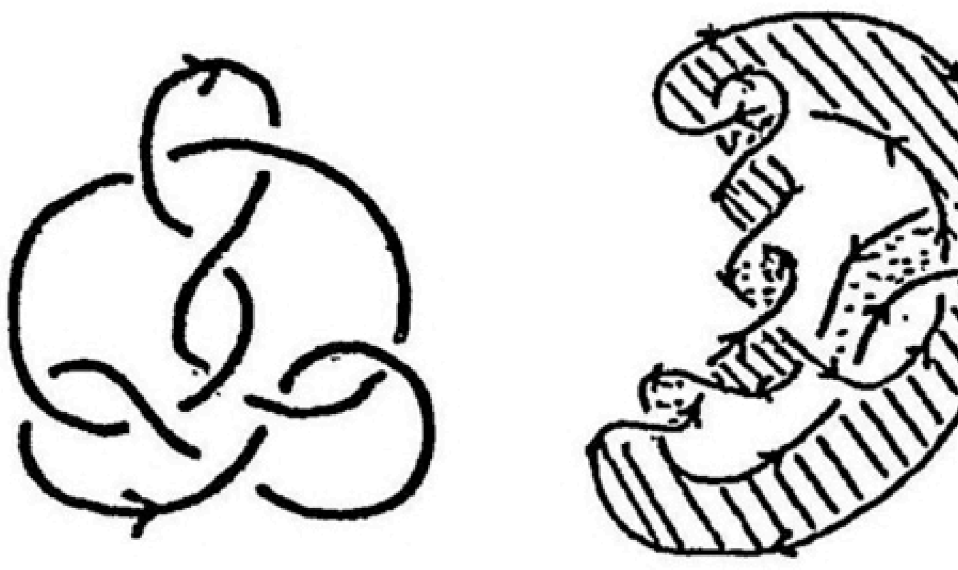
9^2_{23}②



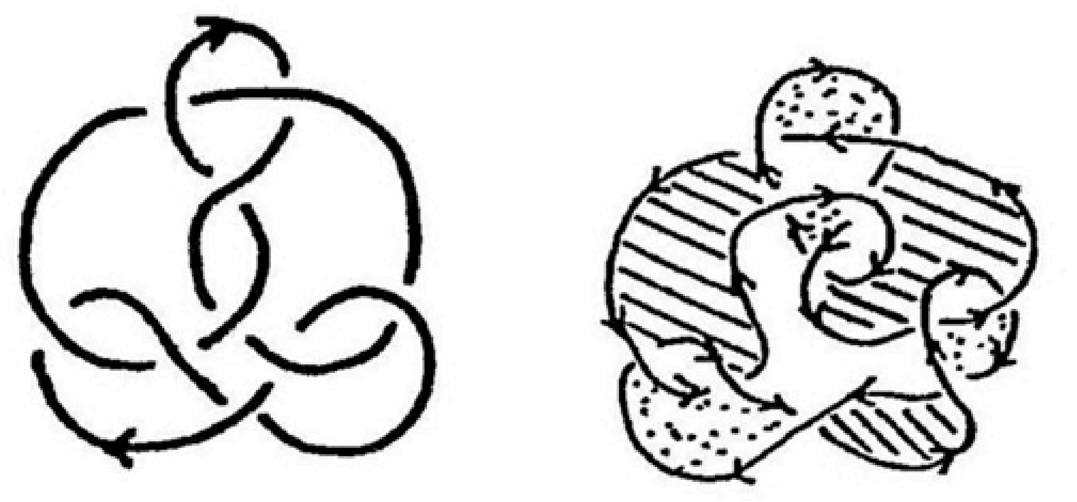
9^2_{24}①



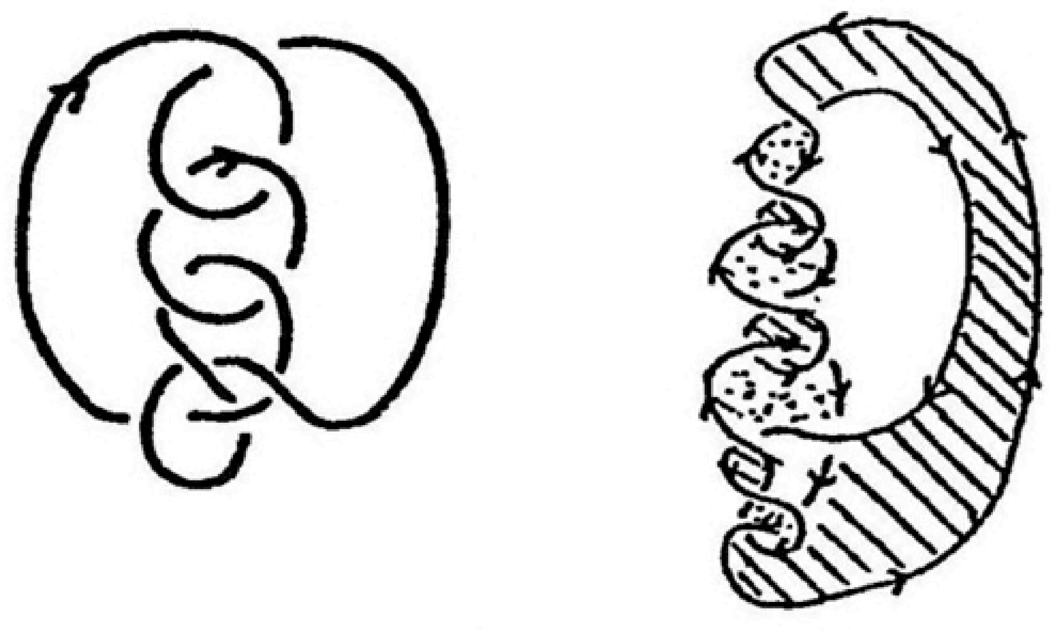
9^2_{24}②



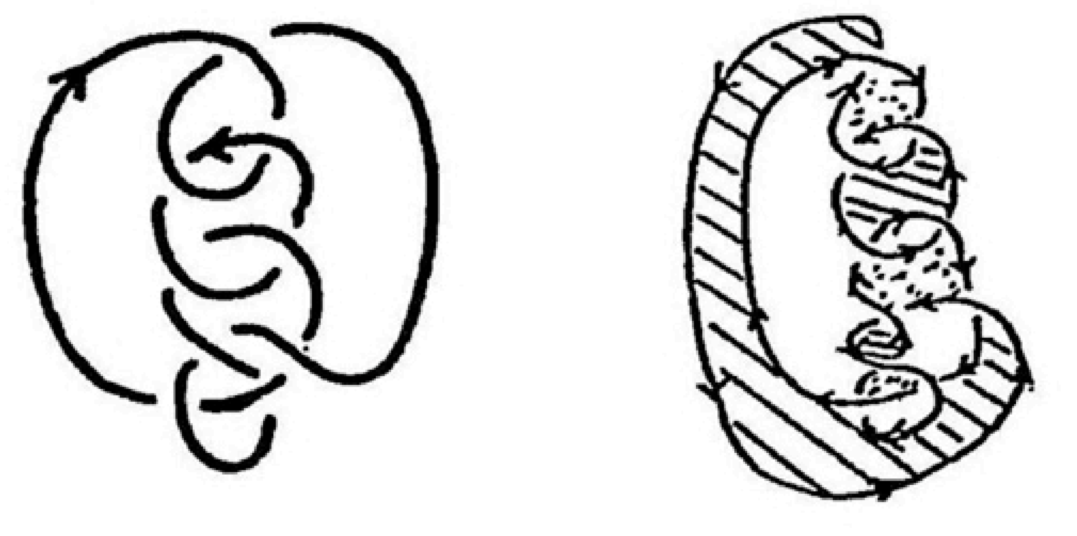
9^2_{25}①



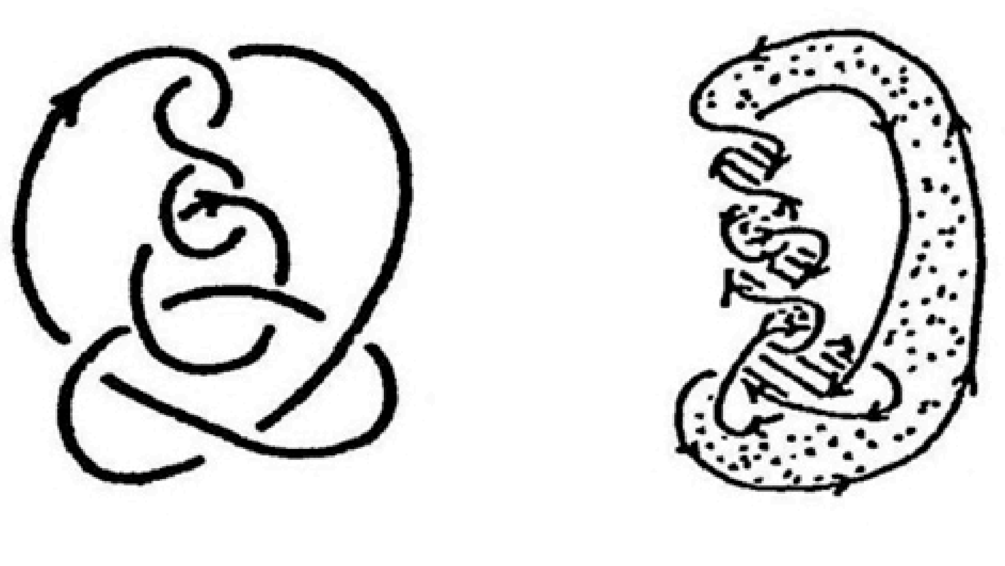
9^2_{25}②



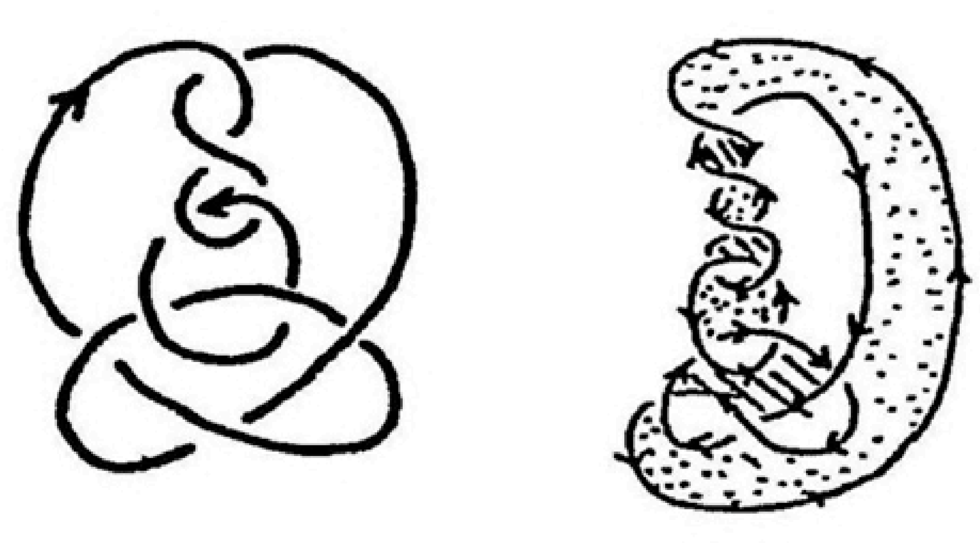
9^2_{26}①



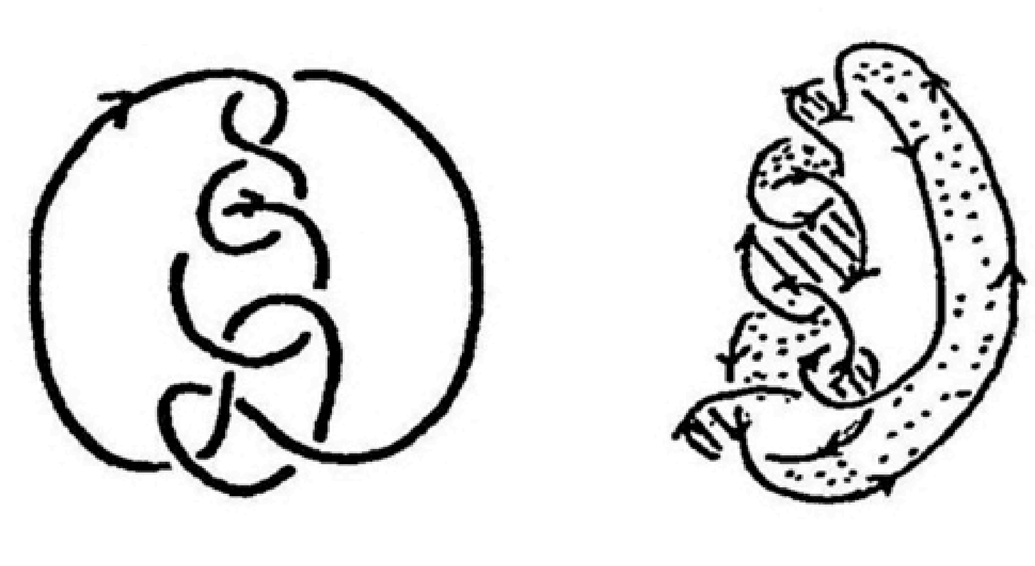
9^2_{26}②



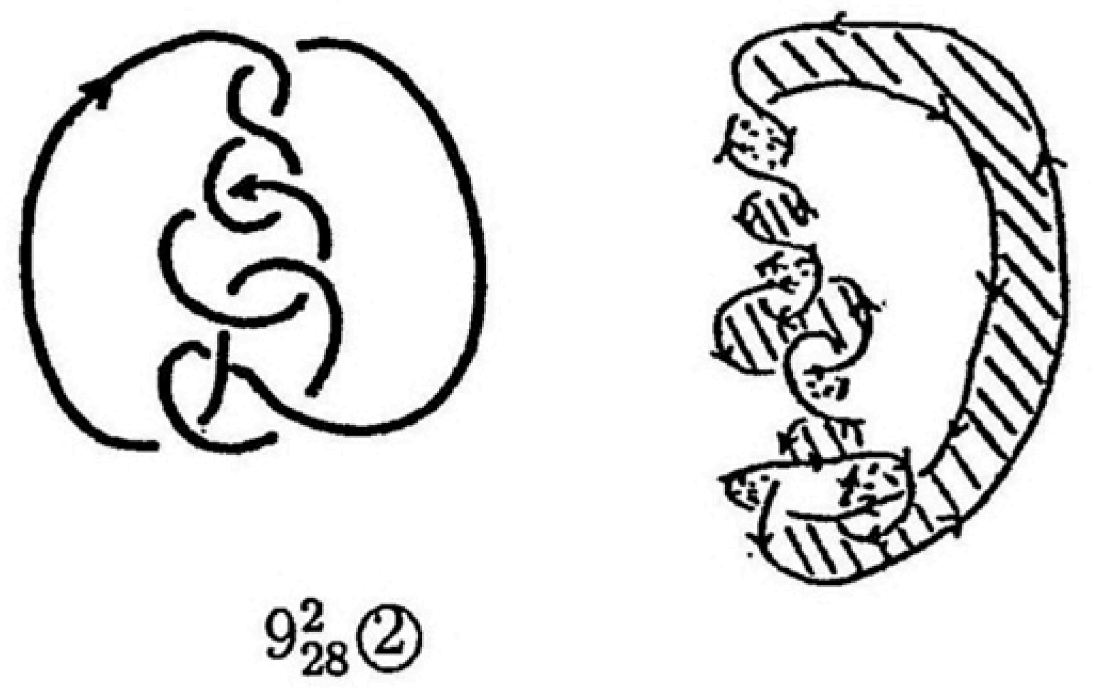
9^2_{27}①



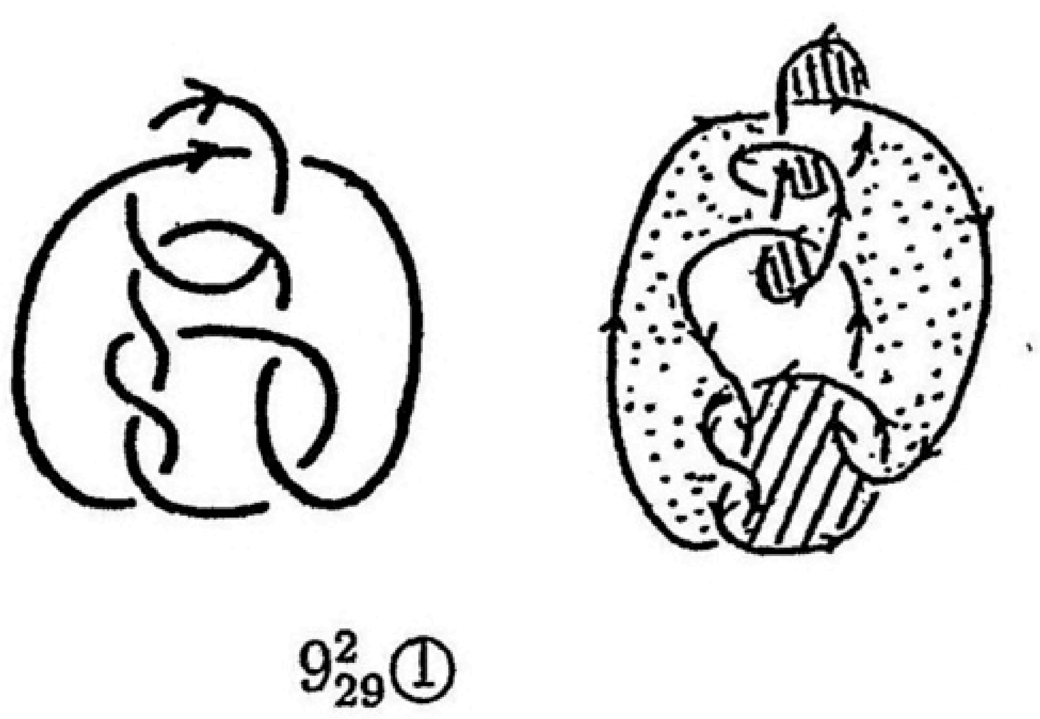
9^2_{27}②



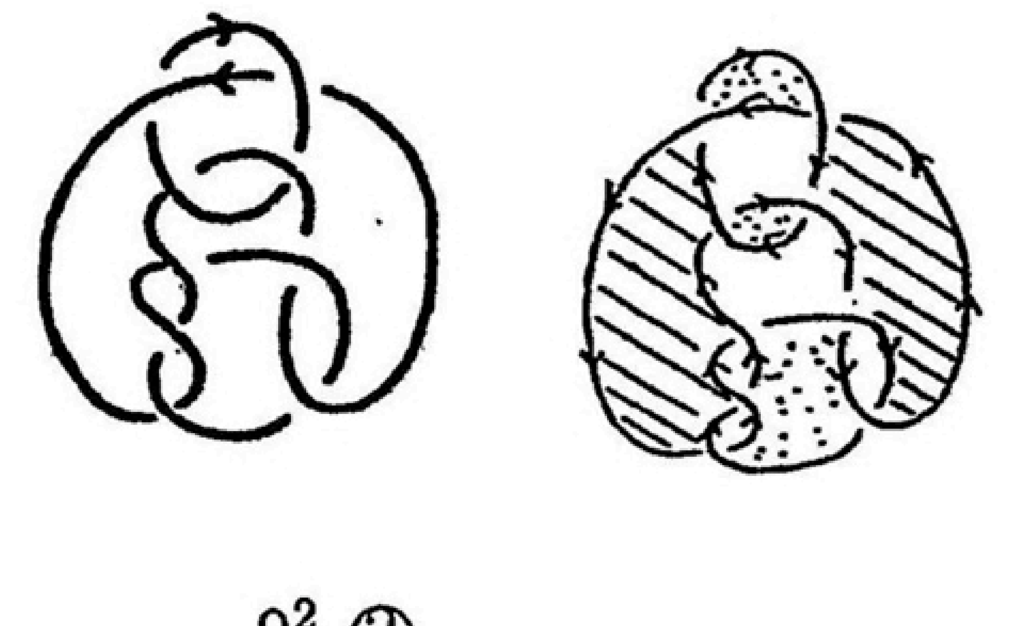
9^2_{28}①



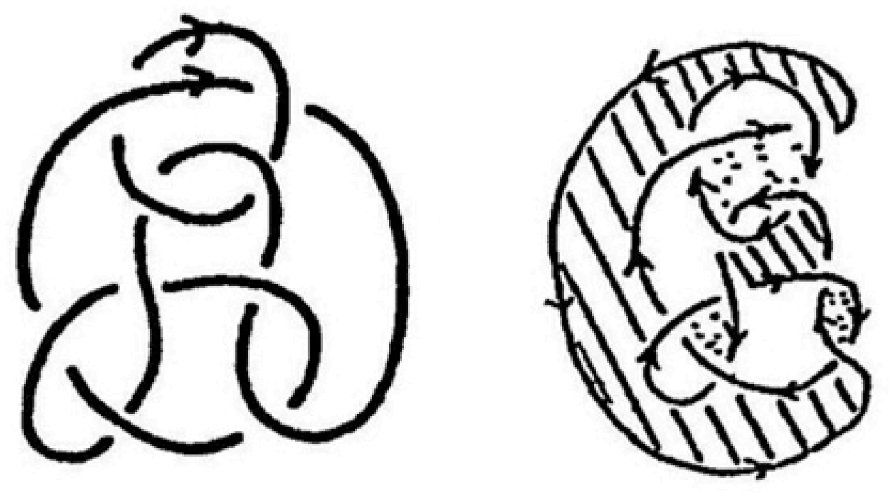
9^2_{28}②



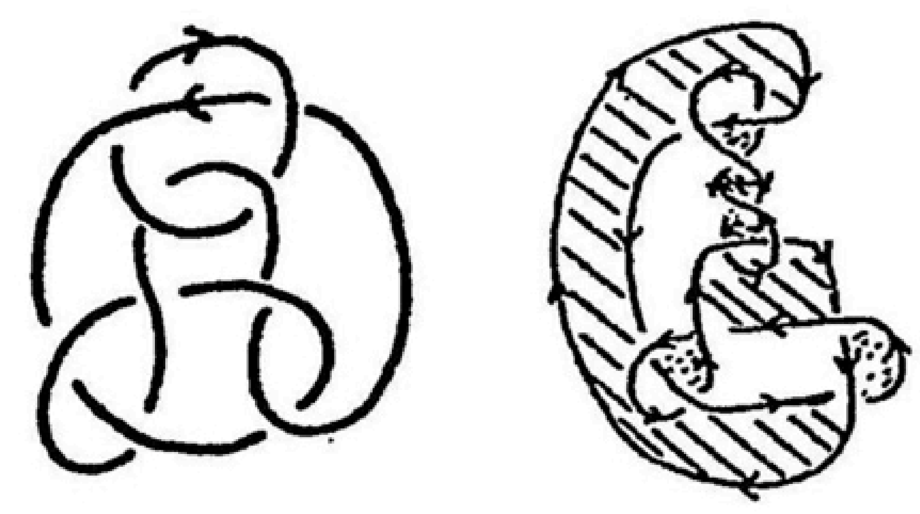
9^2_{29}①



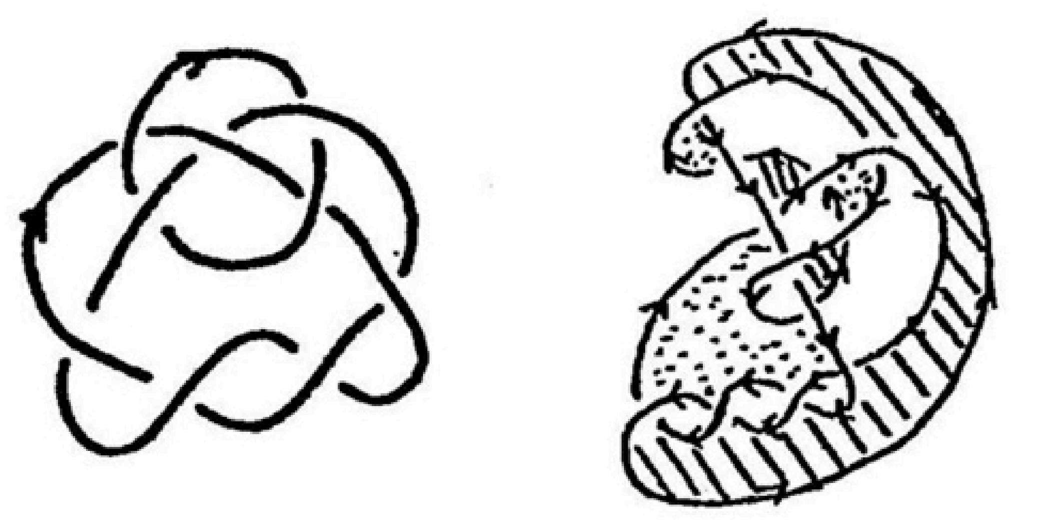
9^2_{29}②



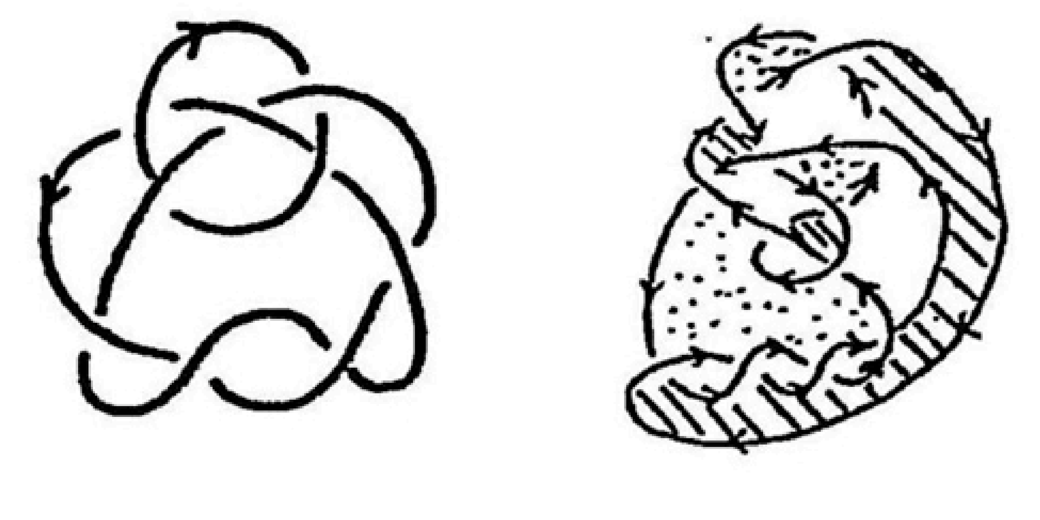
9^2_{30}①



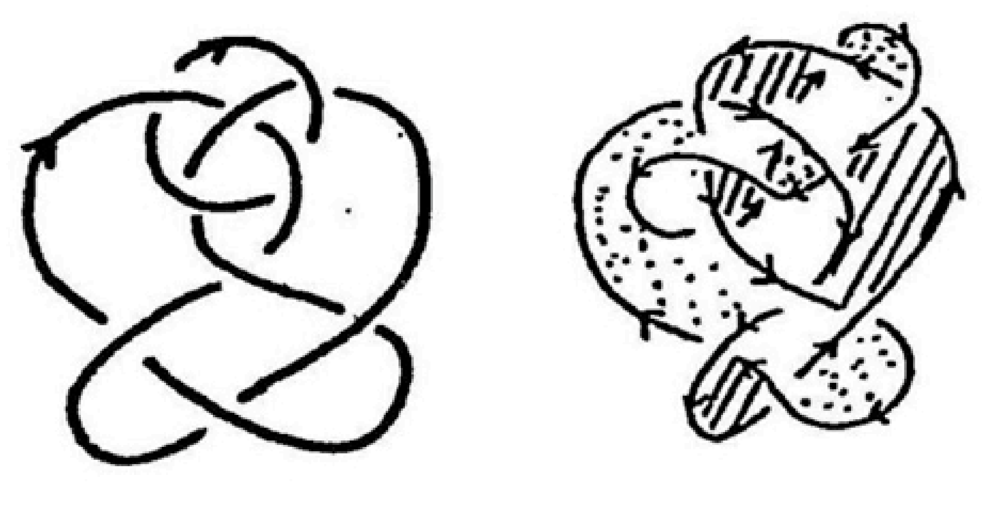
9^2_{30}②



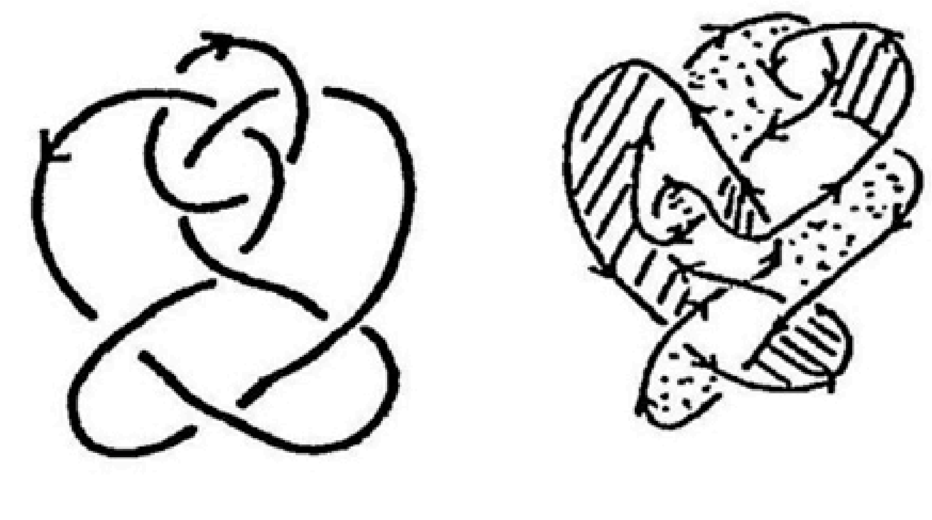
9^2_{31}①



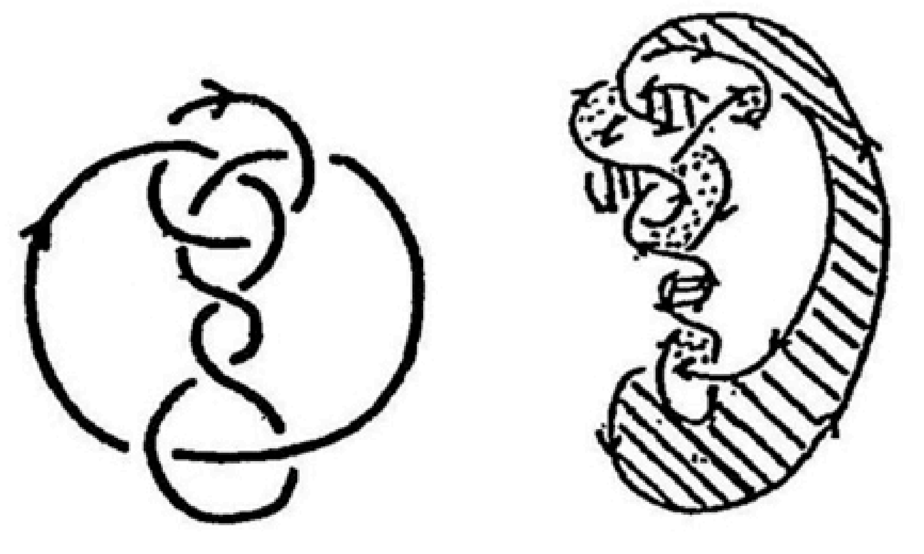
9^2_{31}②



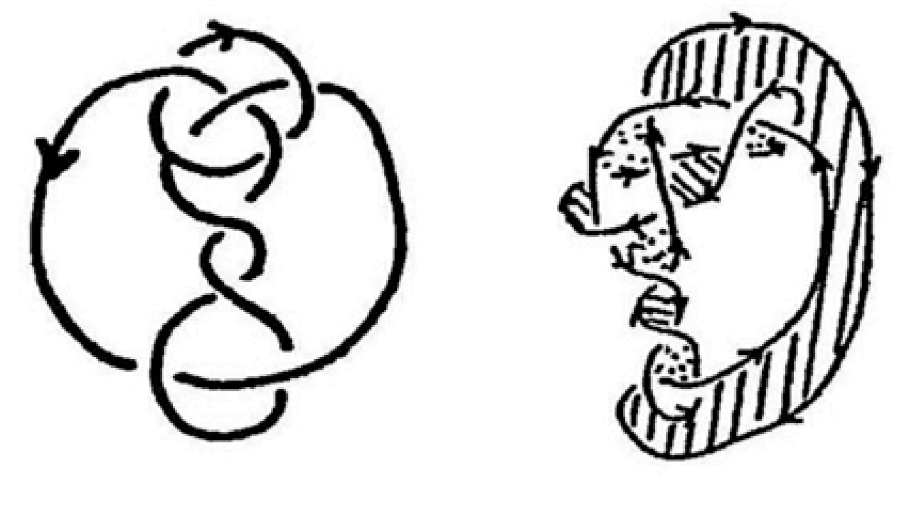
9^2_{32}①



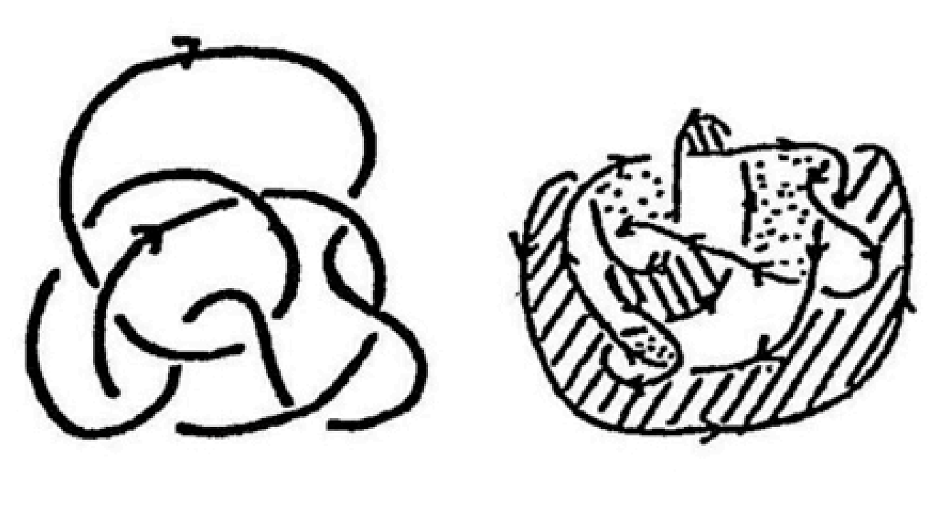
9^2_{32}②



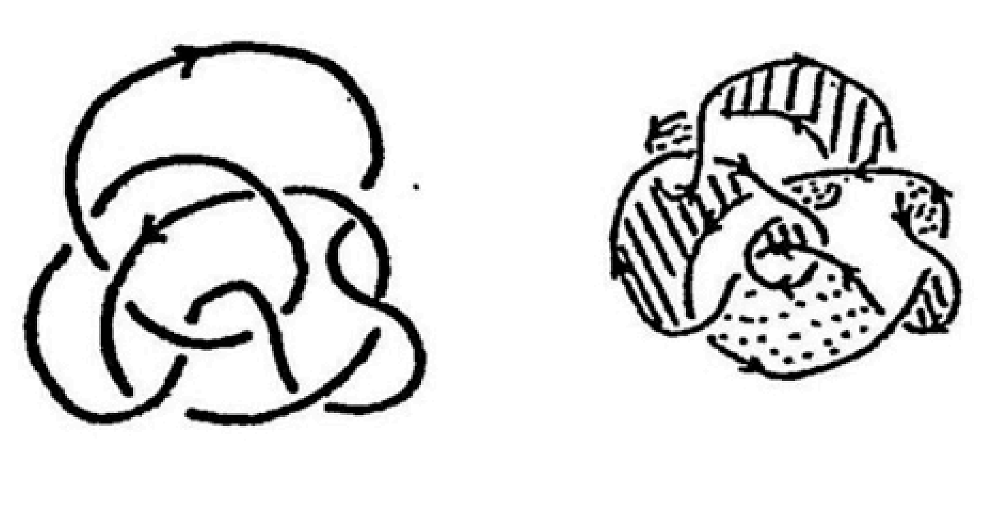
9^2_{33}①



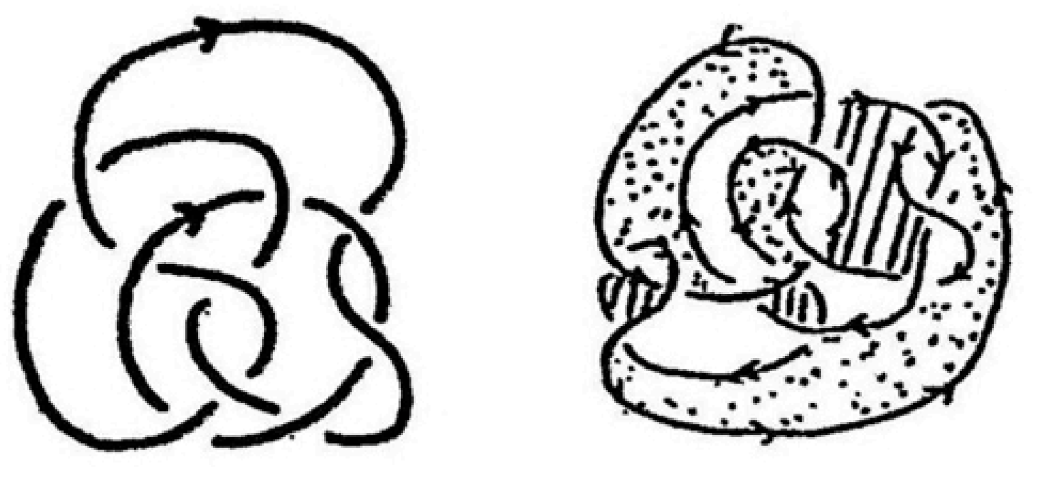
9^2_{33}②



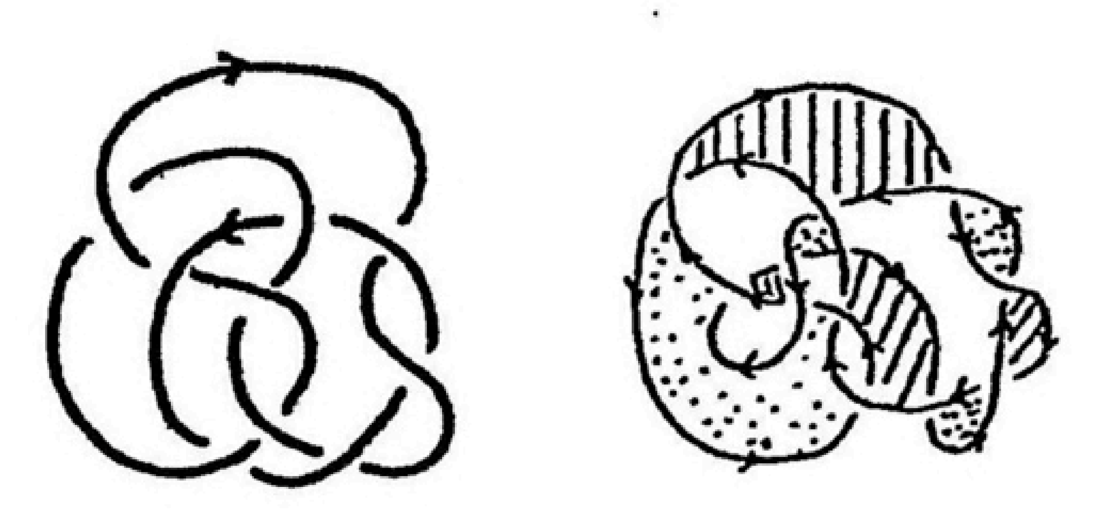
9^2_{34}①



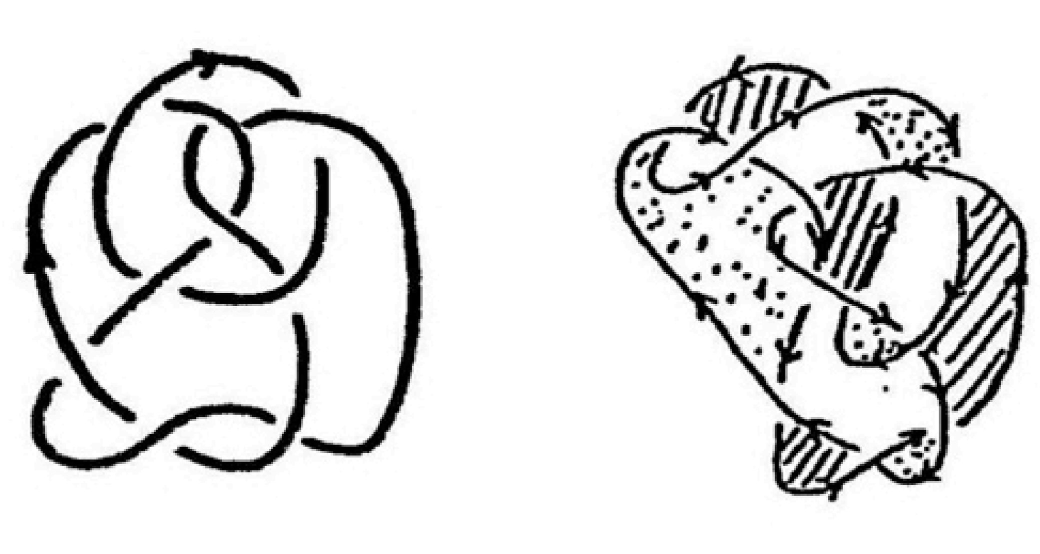
9^2_{34}②



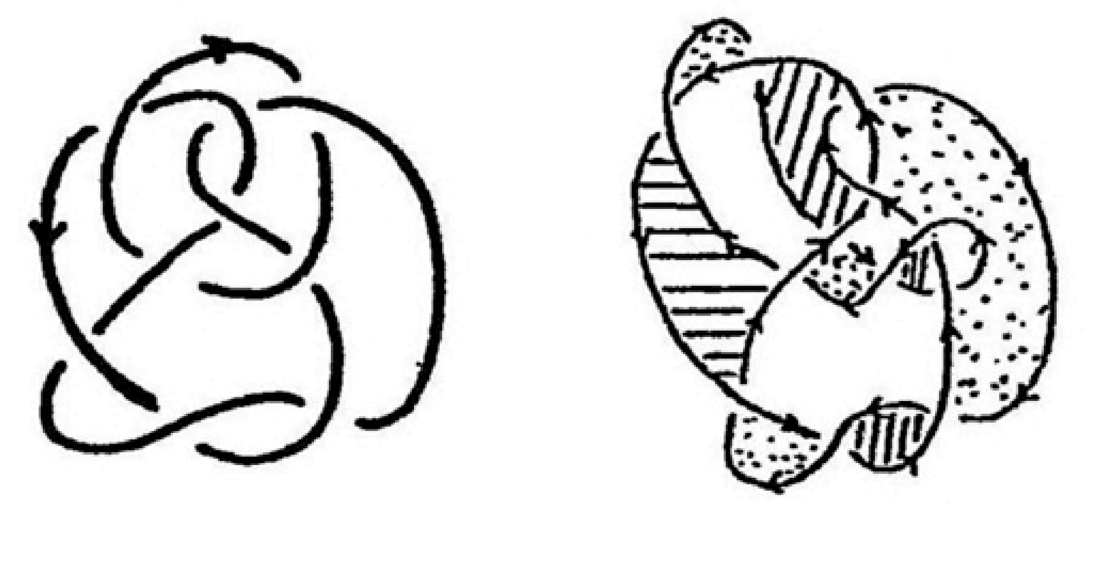
9^{2}_{35} ①



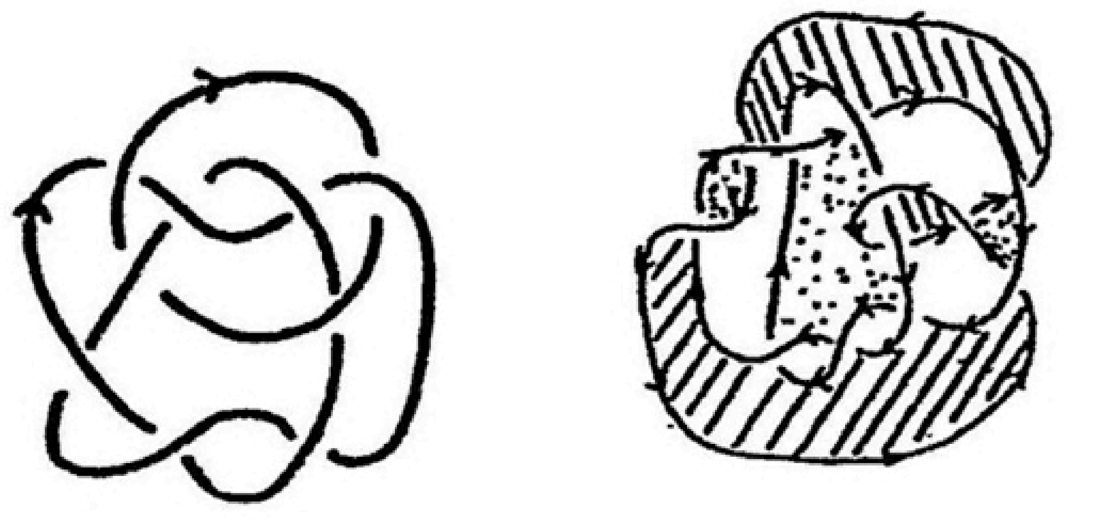
9^{2}_{35} ②



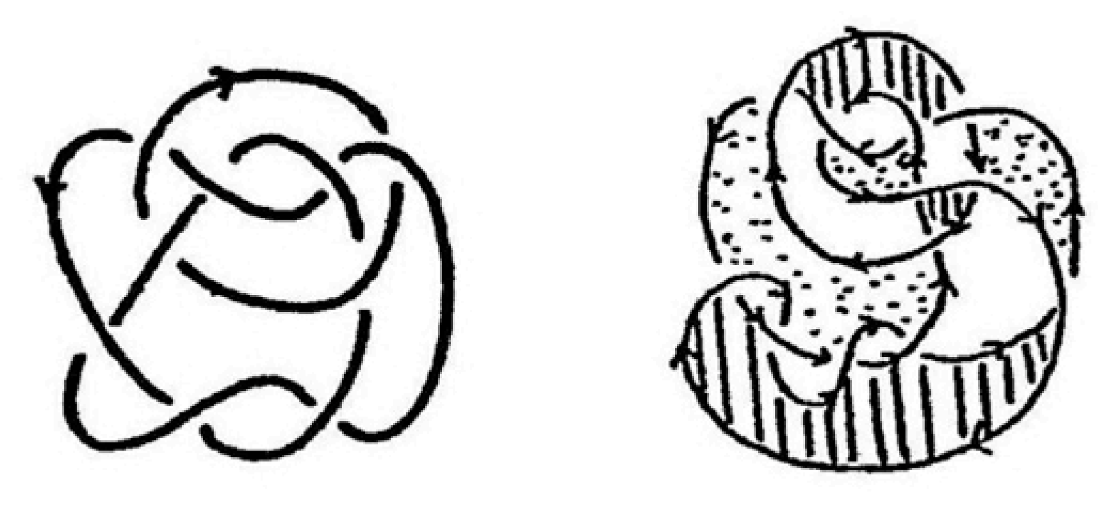
9^{2}_{36} ①



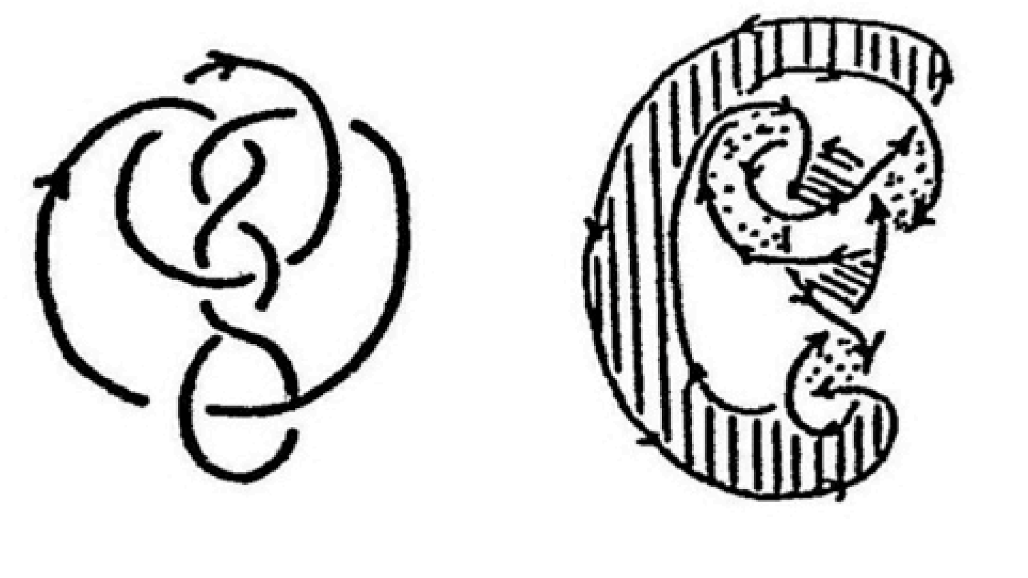
9^{2}_{36} ②



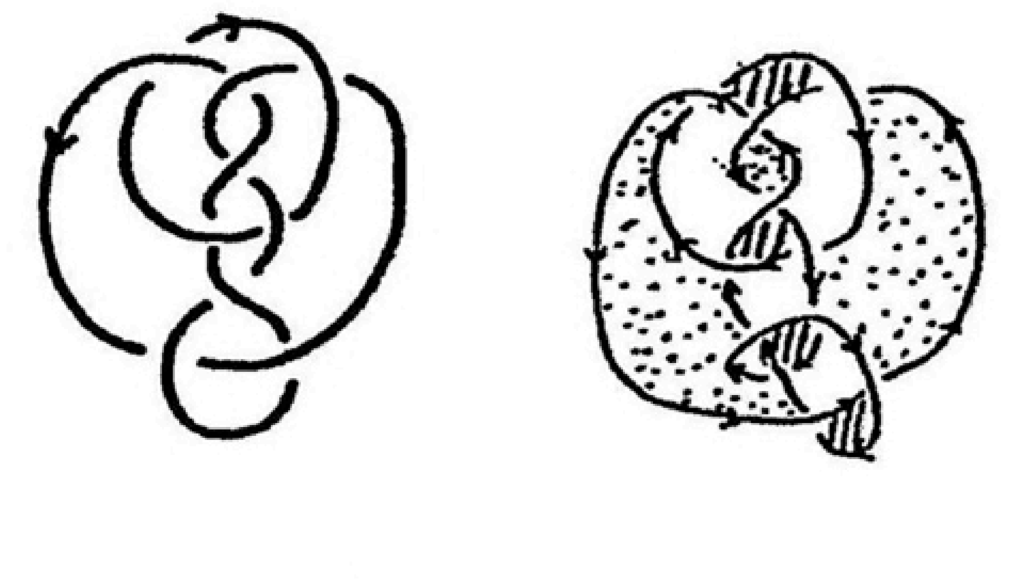
9^{2}_{37} ①



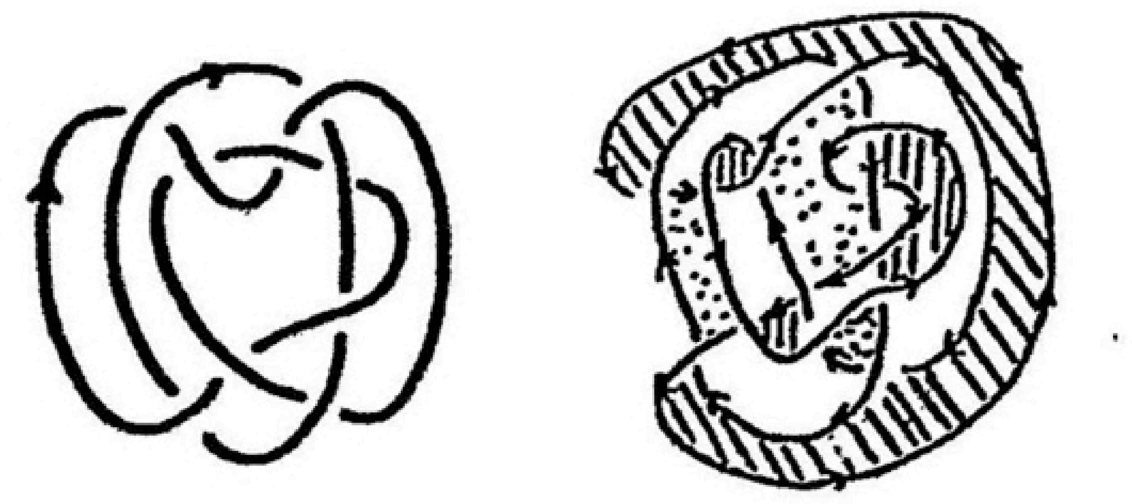
9^{2}_{37} ②



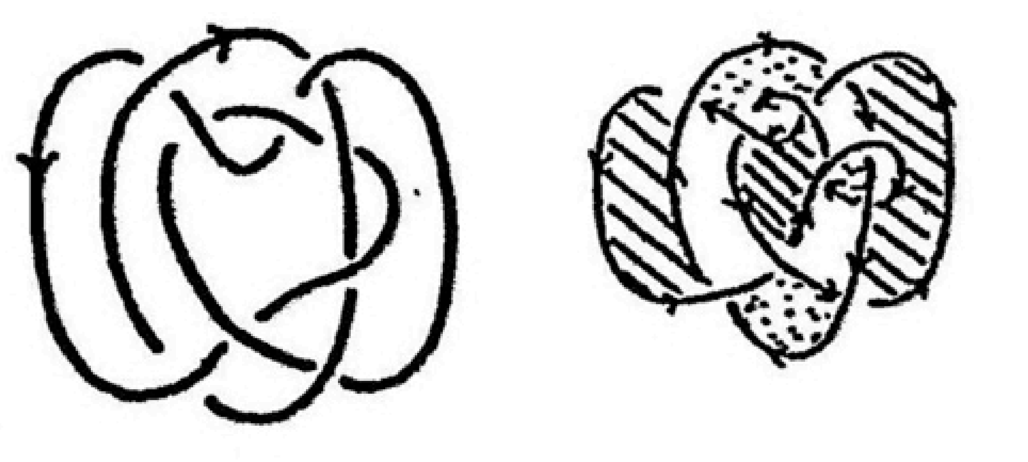
9^{2}_{38} ①



9^{2}_{38} ②

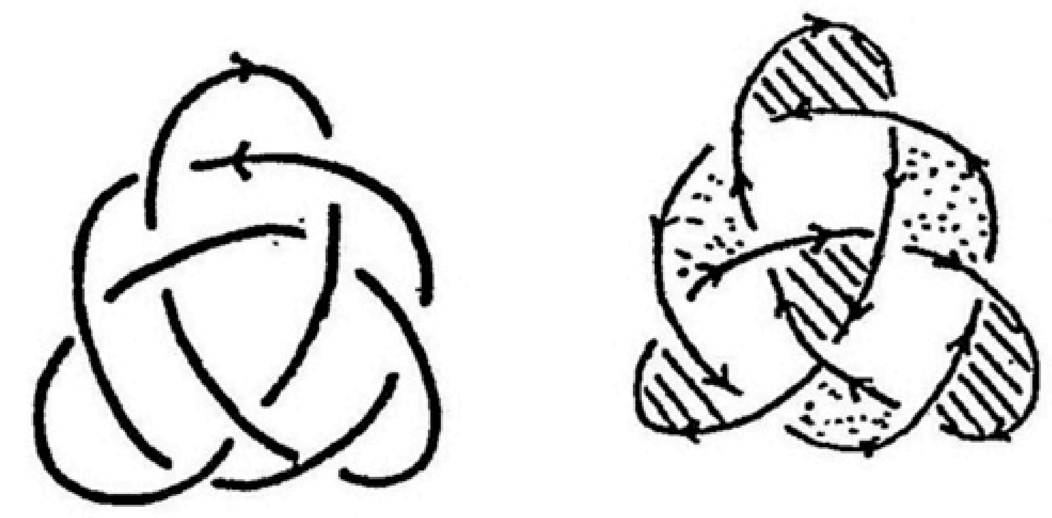


9^{2}_{39} ①

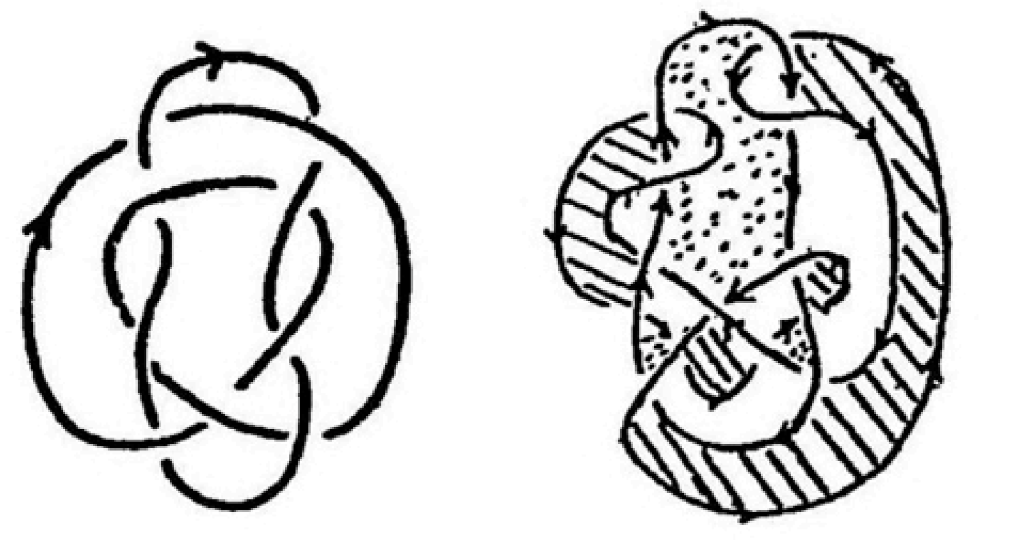


9^{2}_{39} ②

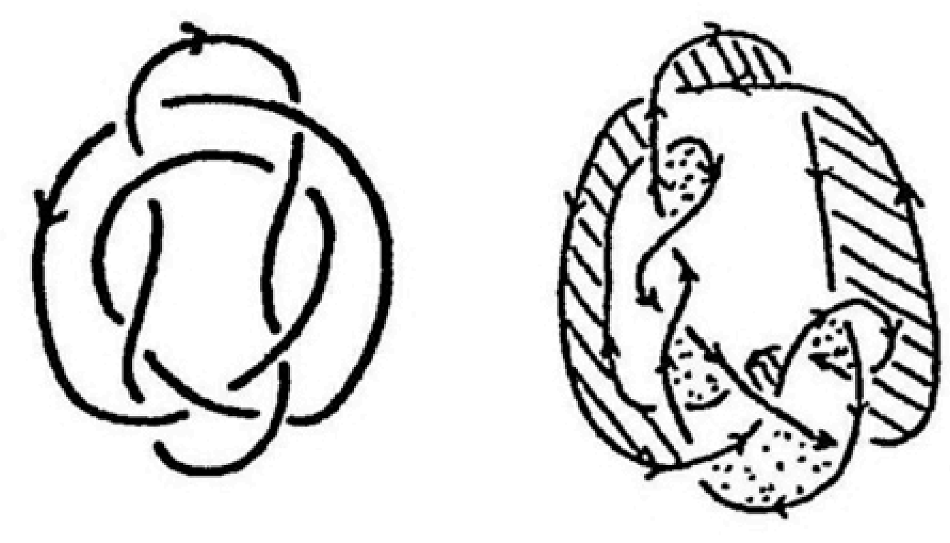
9^2_{40}①



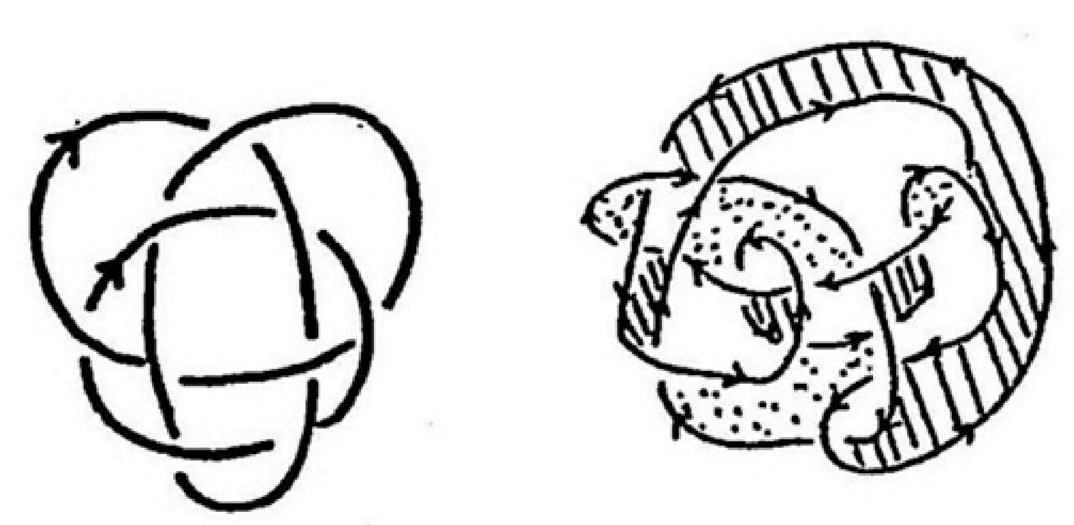
9^2_{40}②



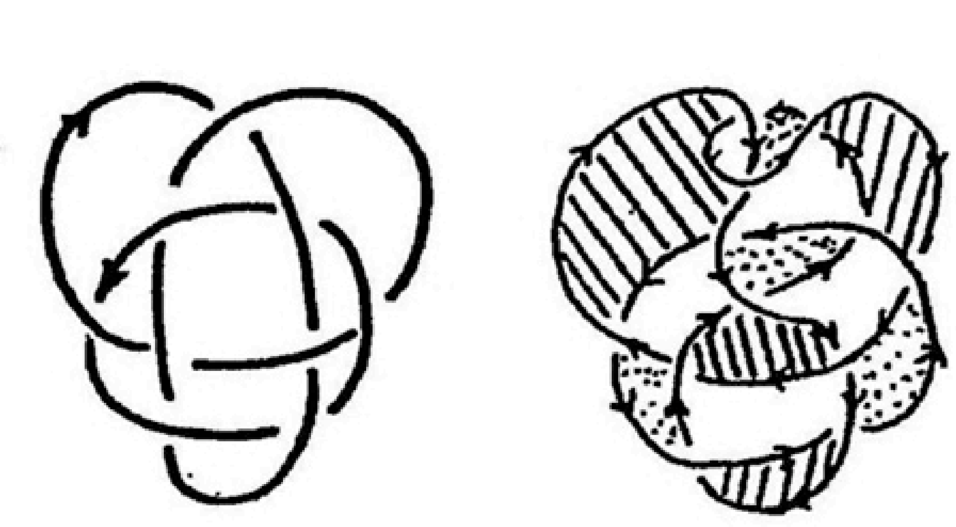
9^2_{41}①



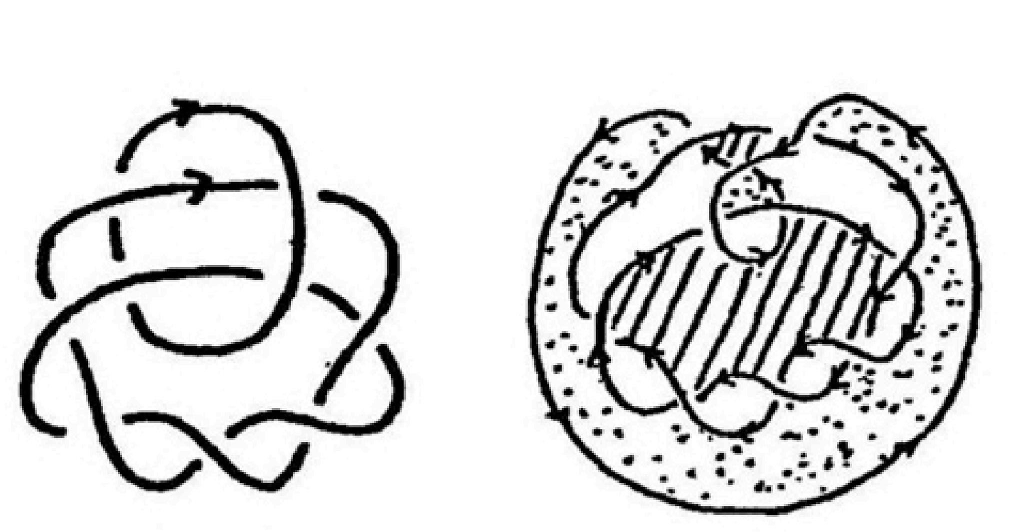
9^2_{41}②



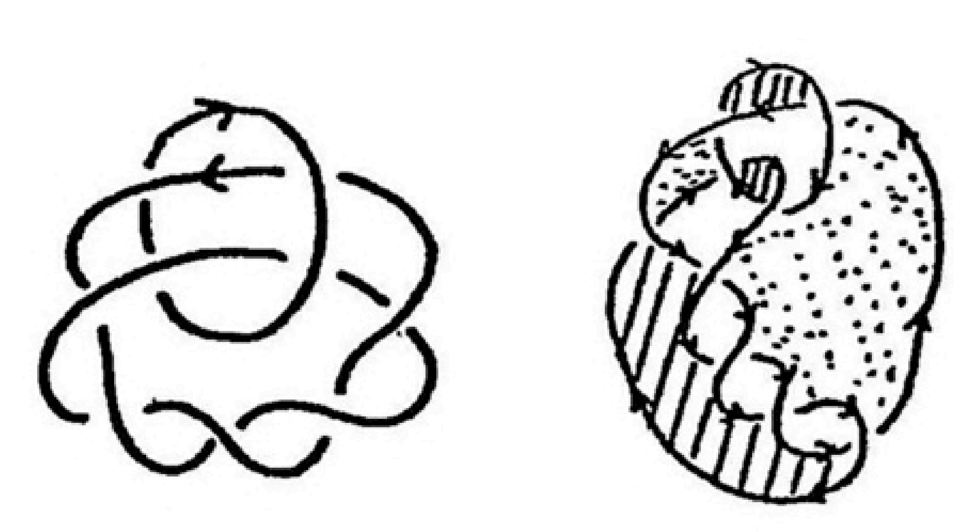
9^2_{42}①



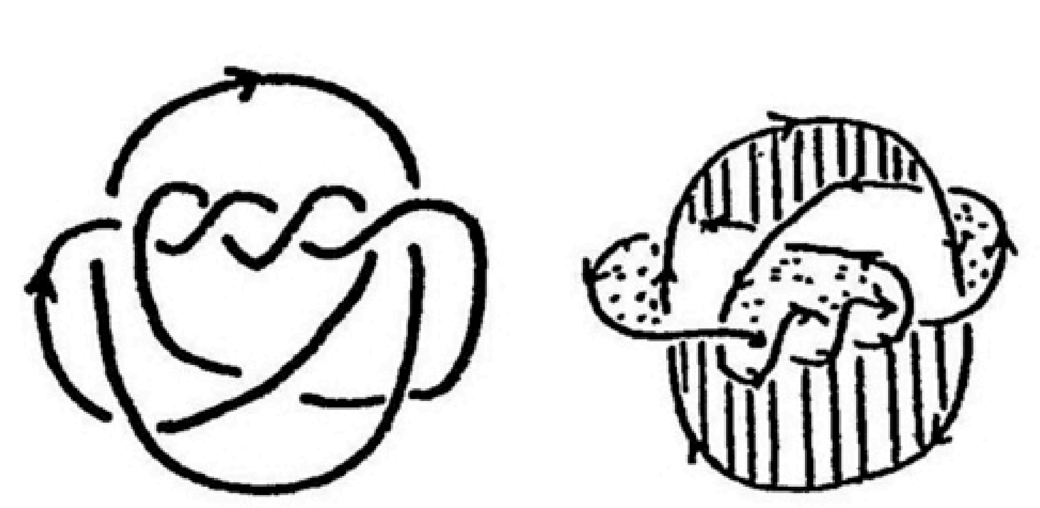
9^2_{42}②



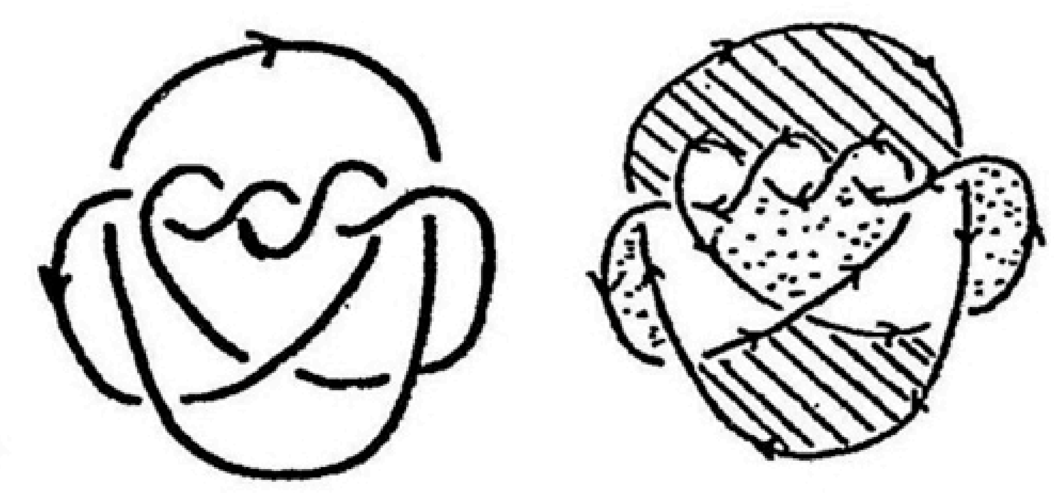
9^2_{43}①



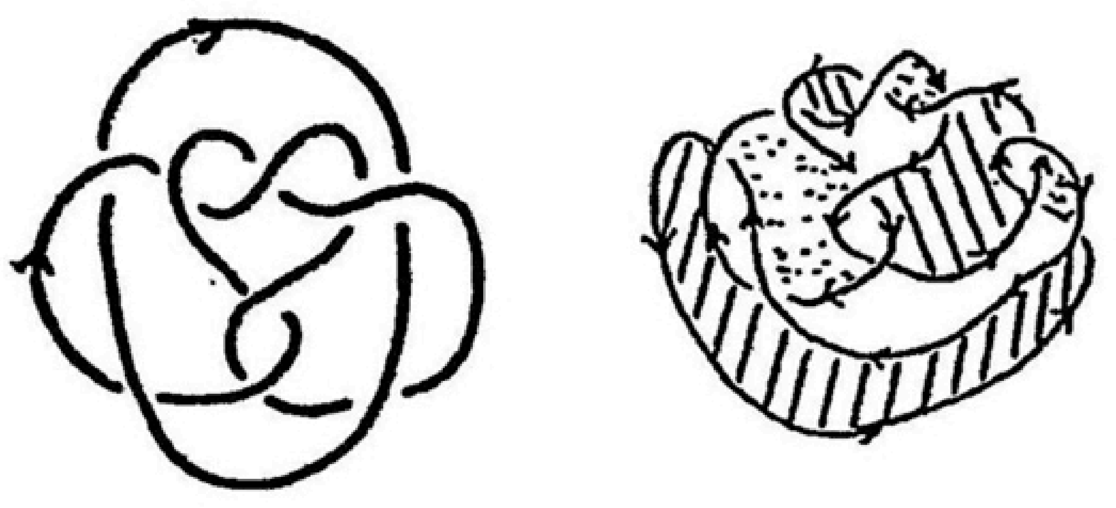
9^2_{43}②



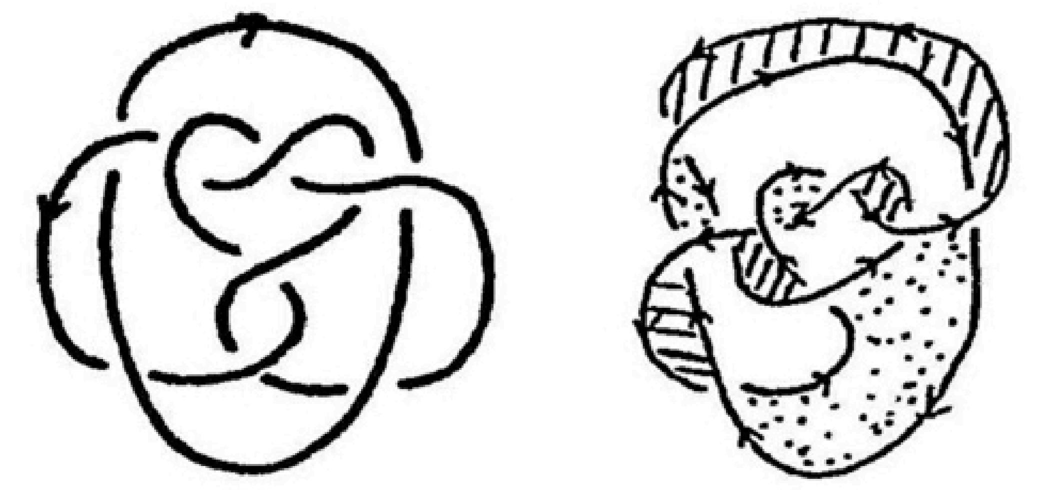
9^2_{44}①



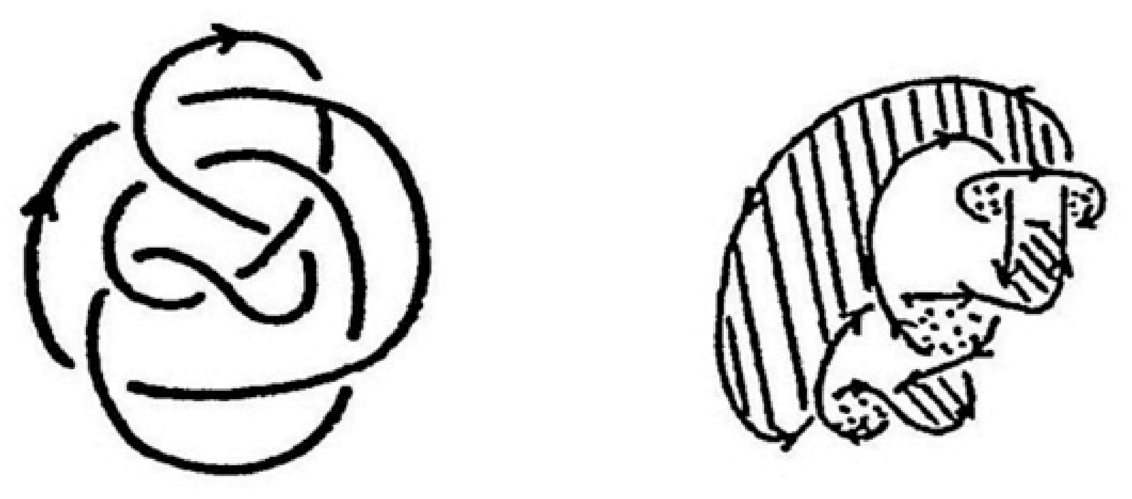
9^2_{44}②



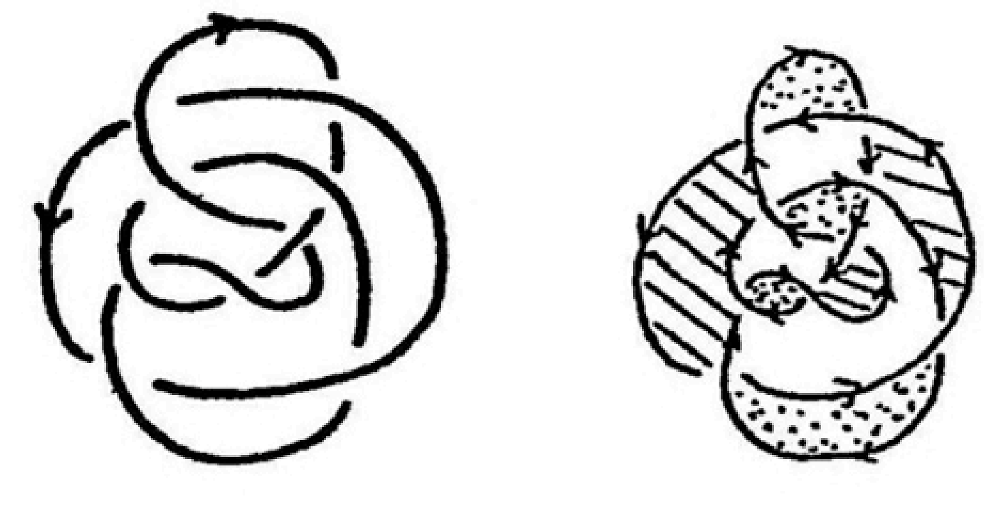
9^2_{45}①



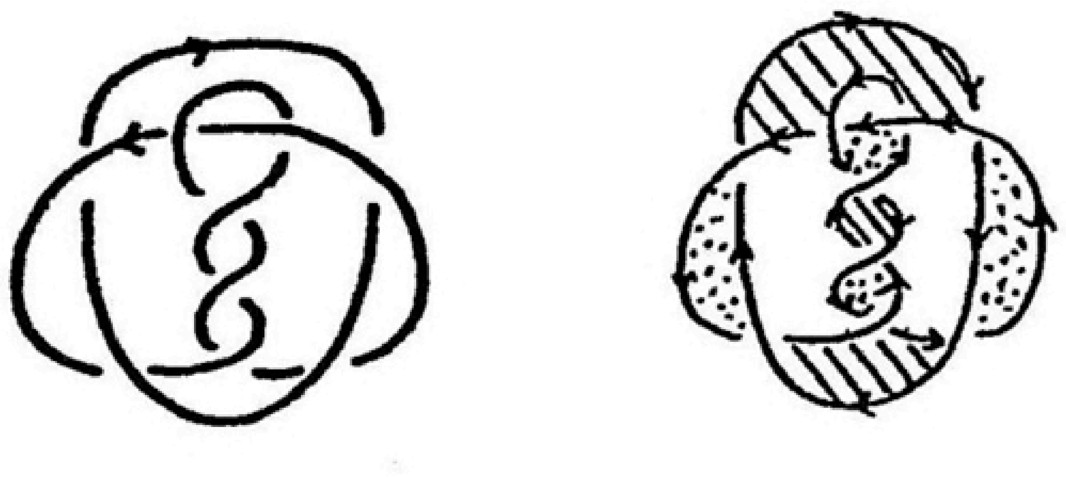
9^2_{45}②



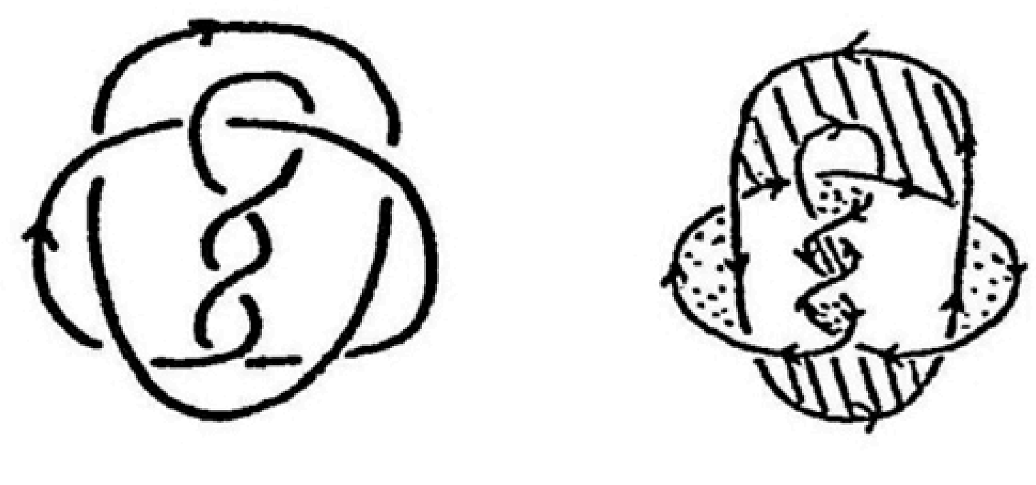
9^2_{46}①



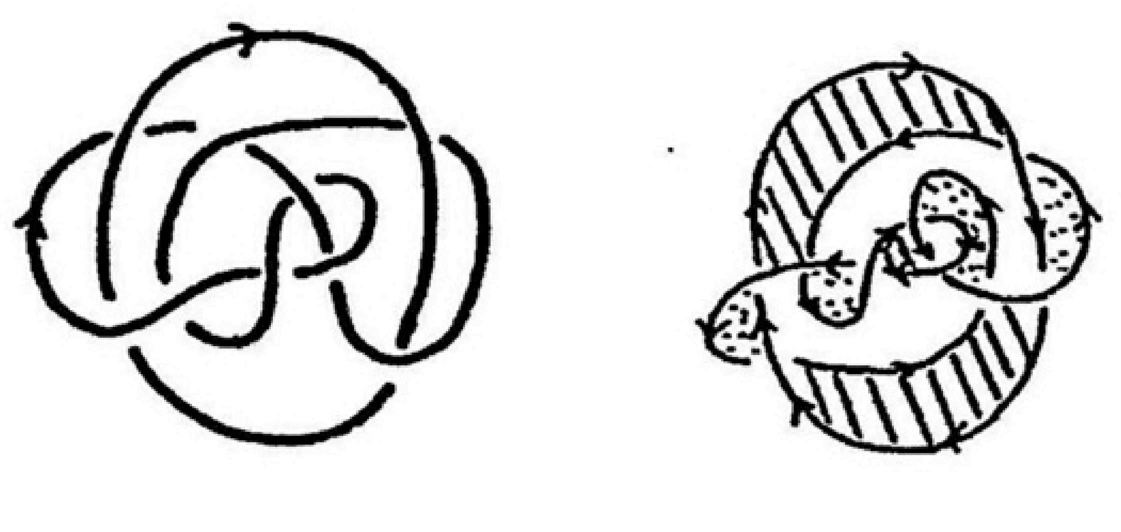
9^2_{46}②



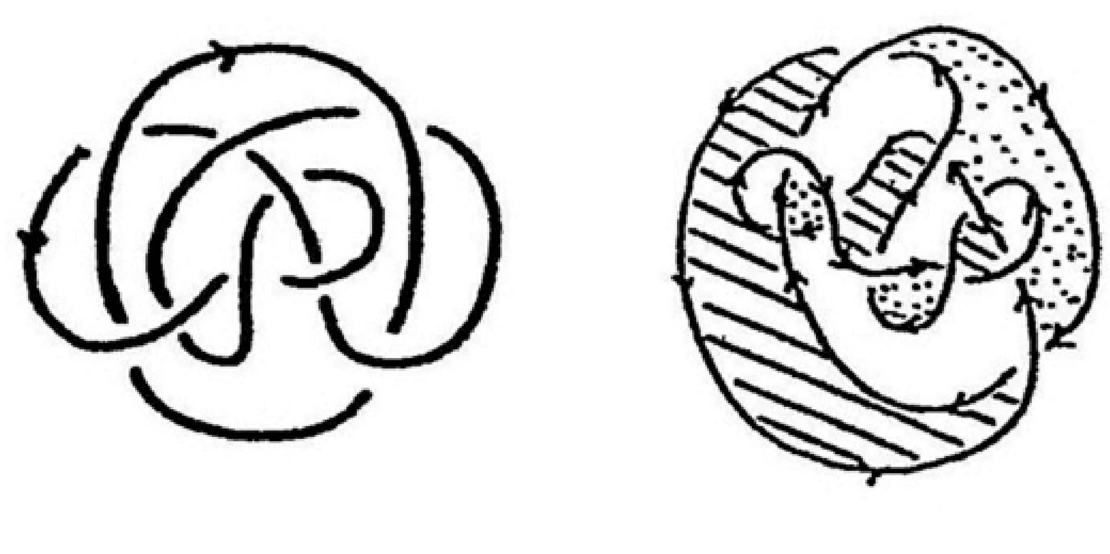
9^2_{47}①



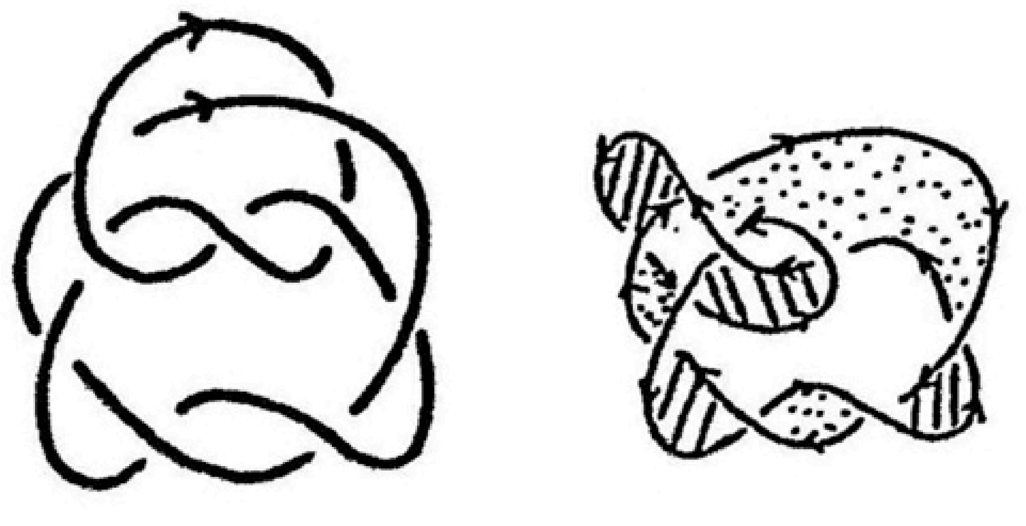
9^2_{47}②



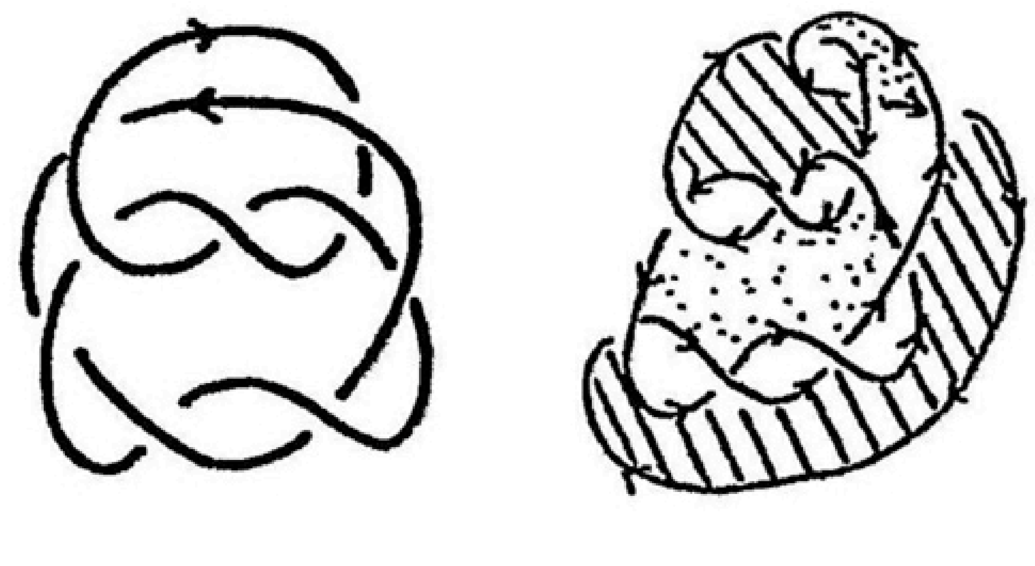
9^2_{48}①



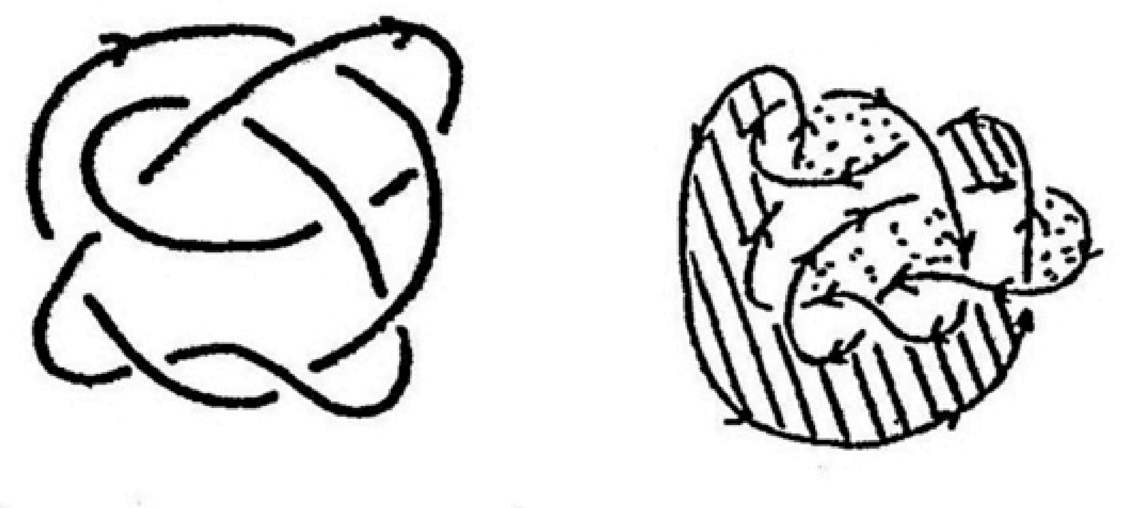
9^2_{48}②



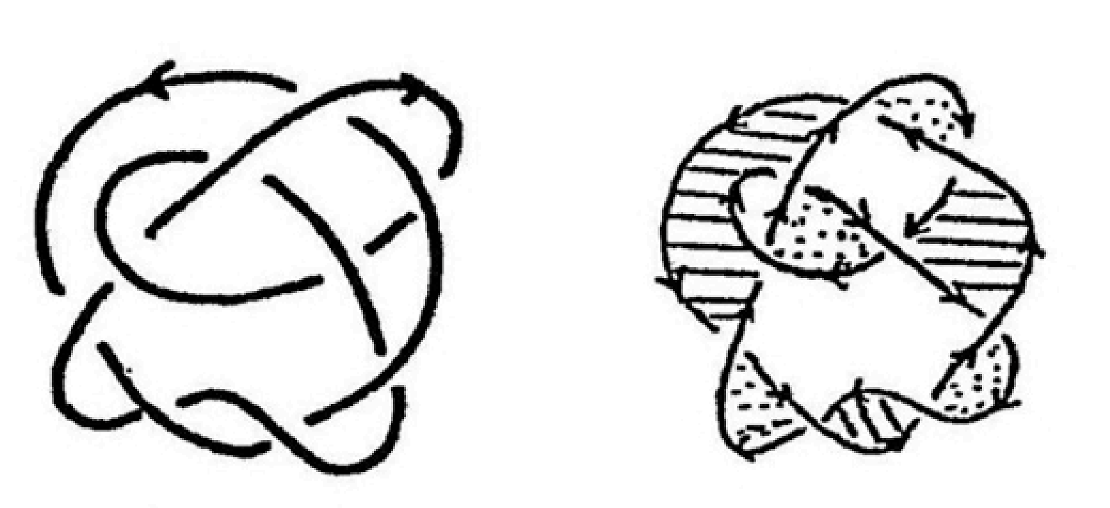
9^2_{49}①



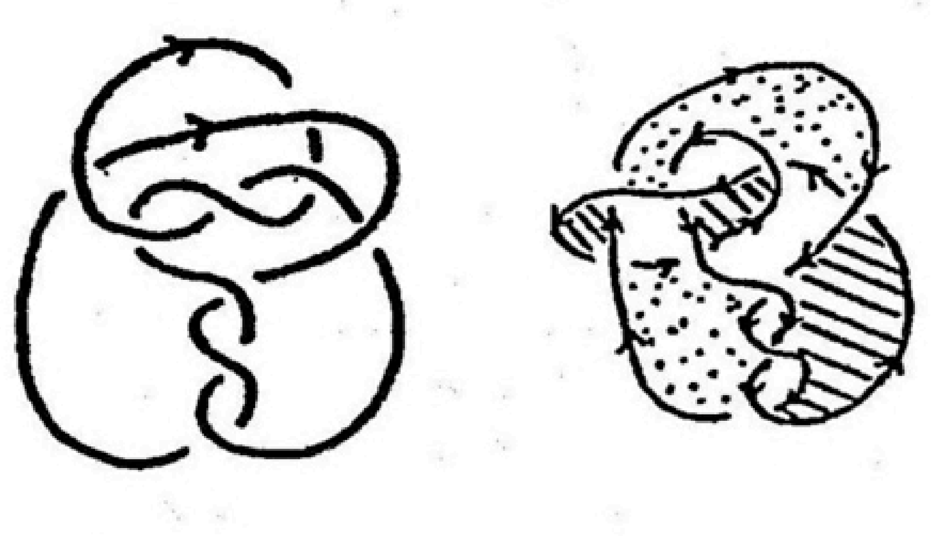
9^2_{49}②



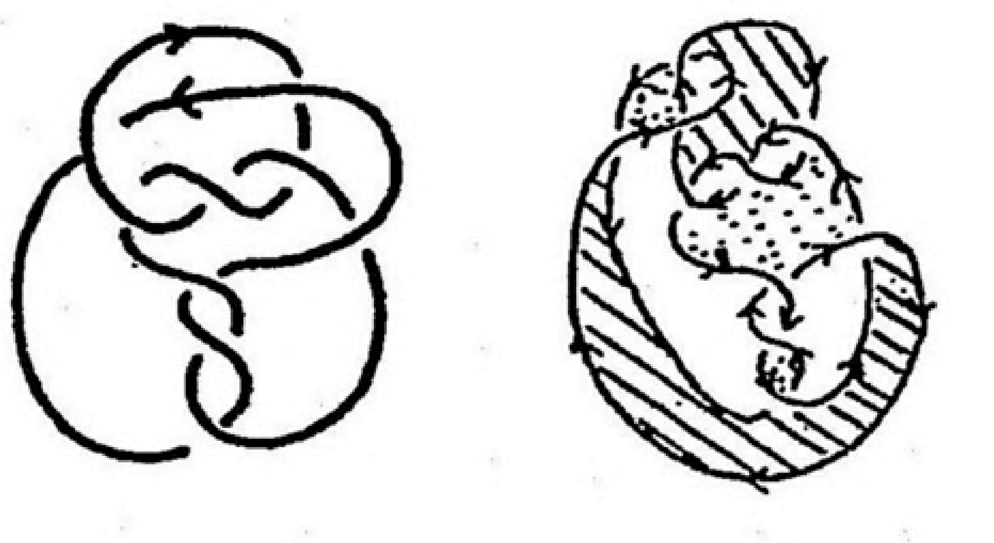
9^2_{50}①



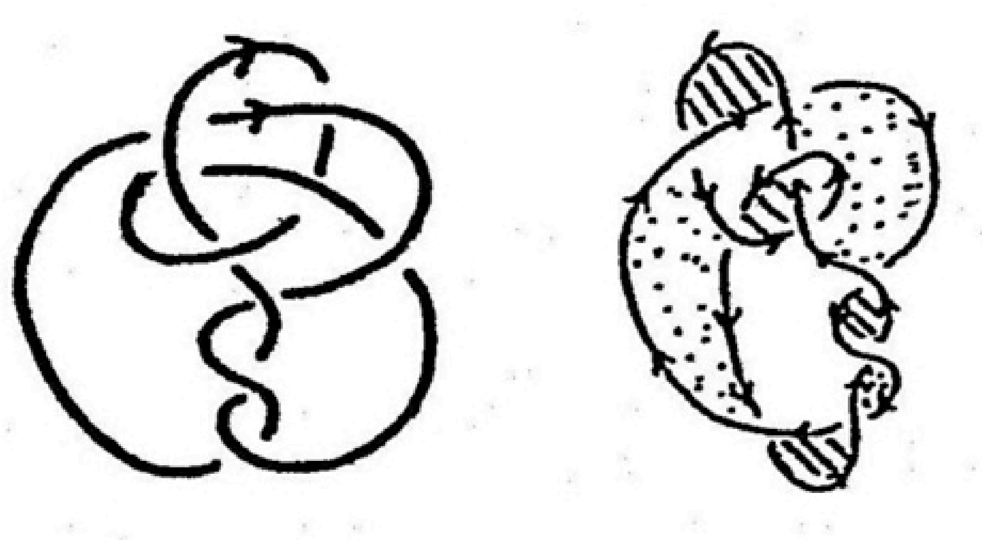
9^2_{50}②



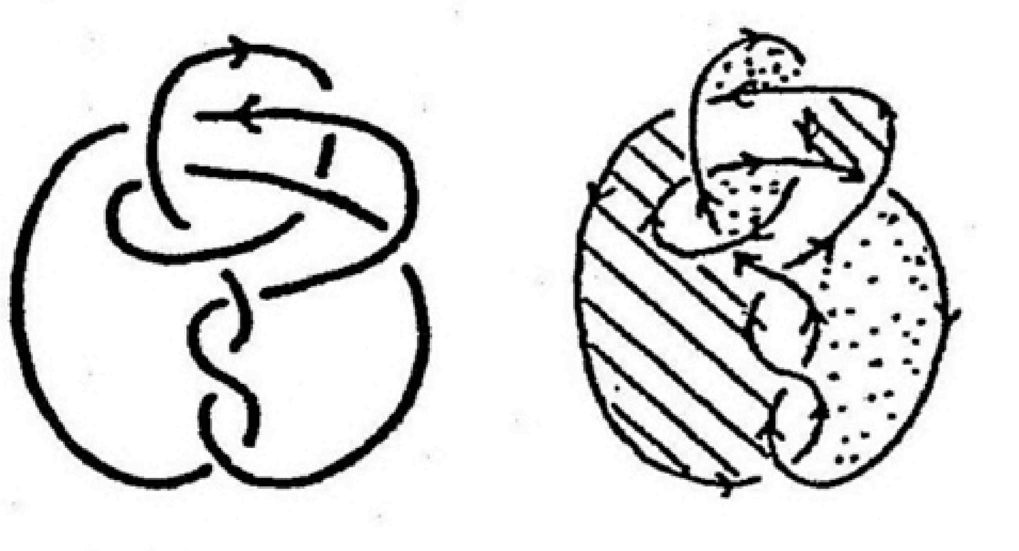
9^2_{51}①



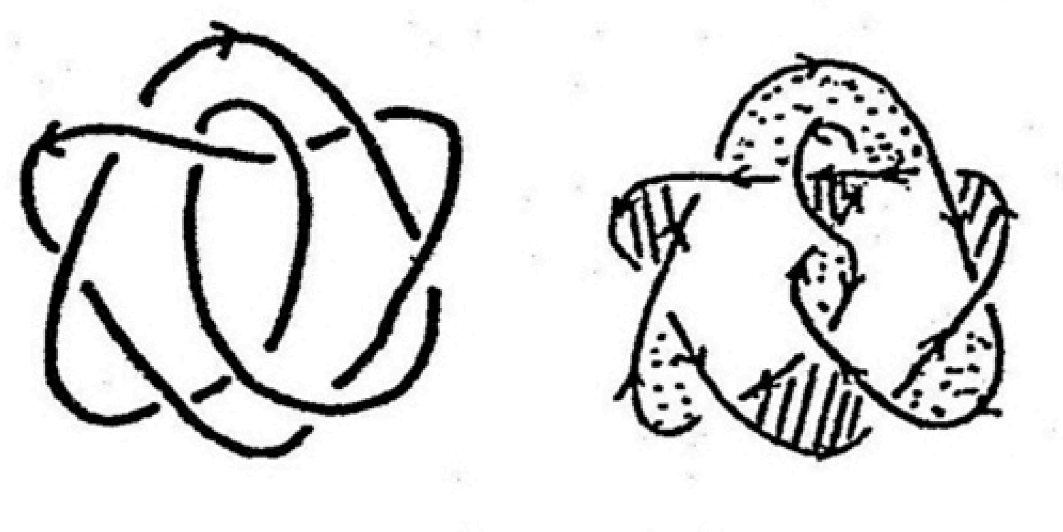
9^2_{51}②



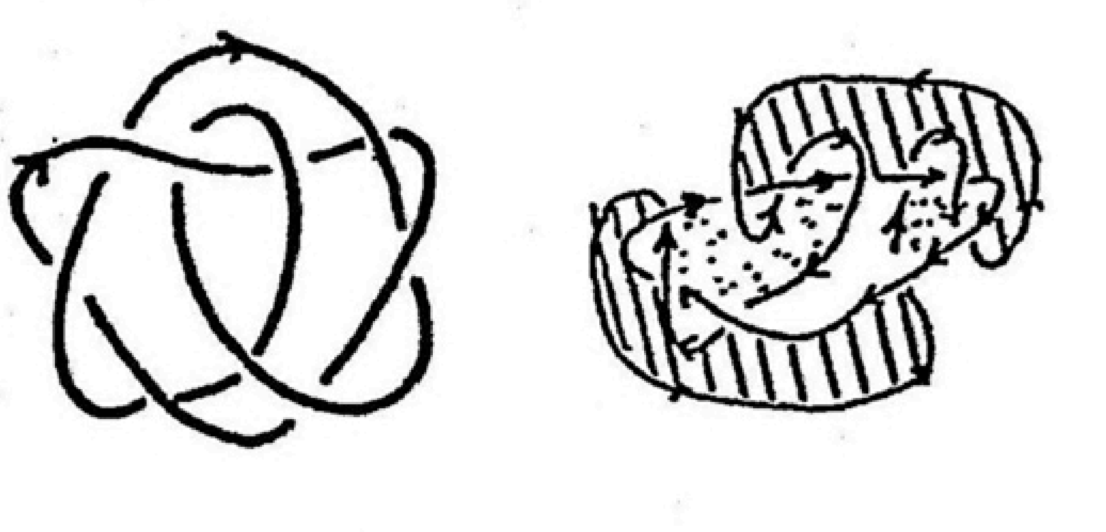
9^2_{52}①



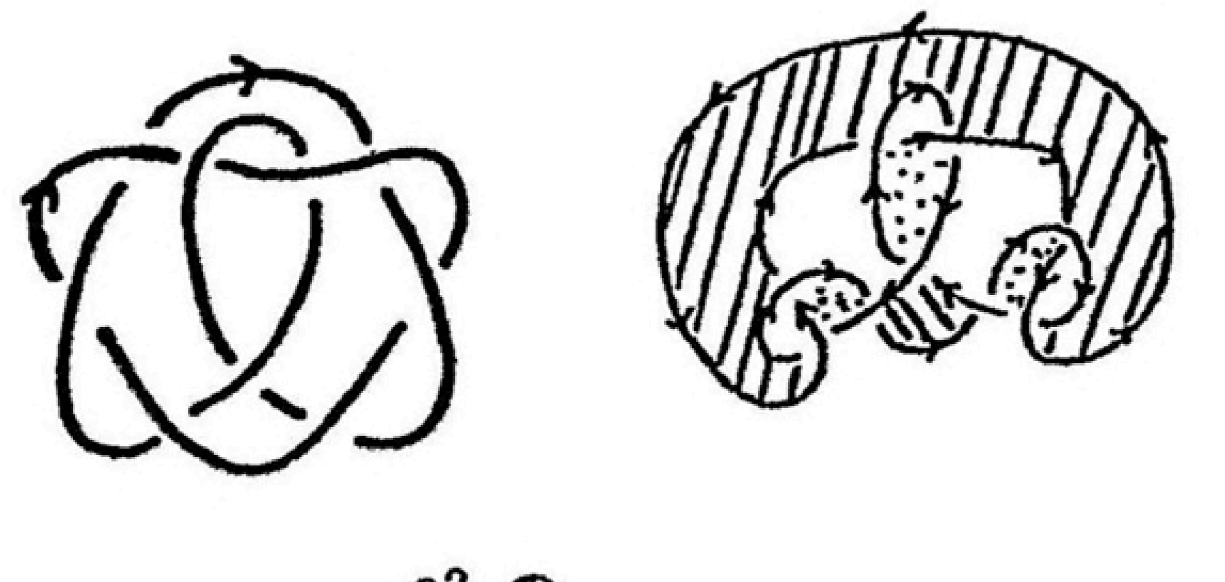
9^2_{52}②



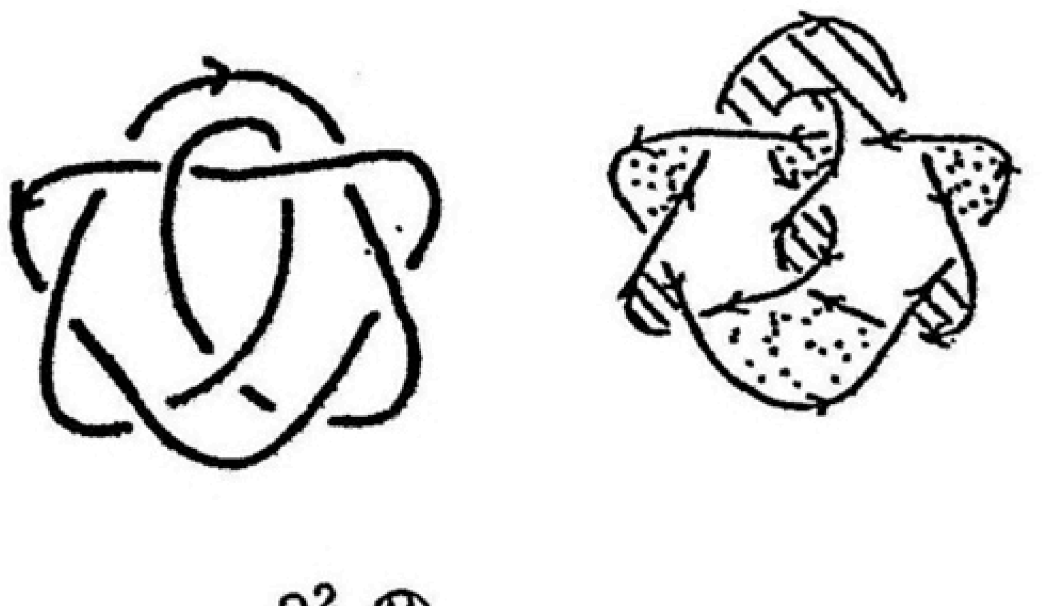
9^2_{53}①



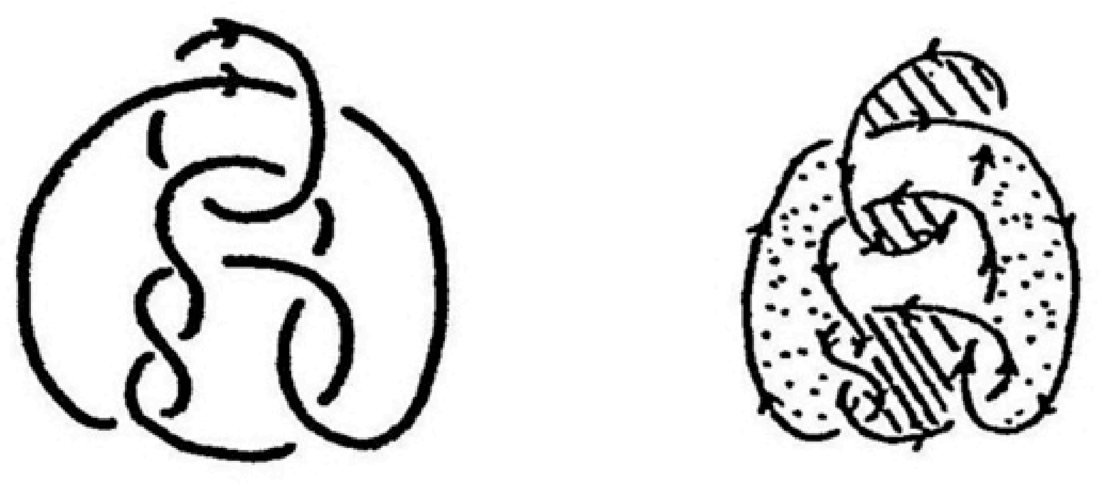
9^2_{53}②



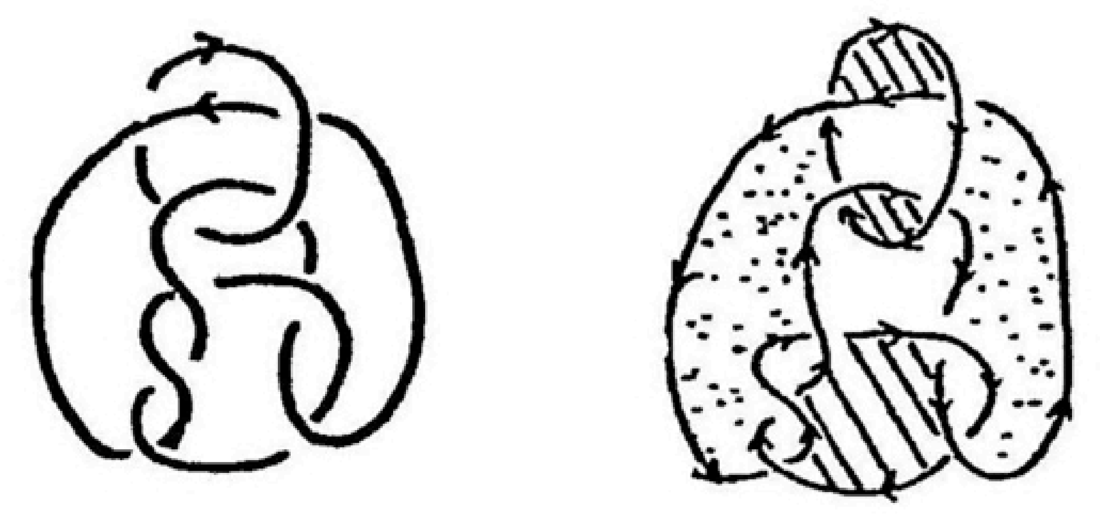
9^2_{54}①



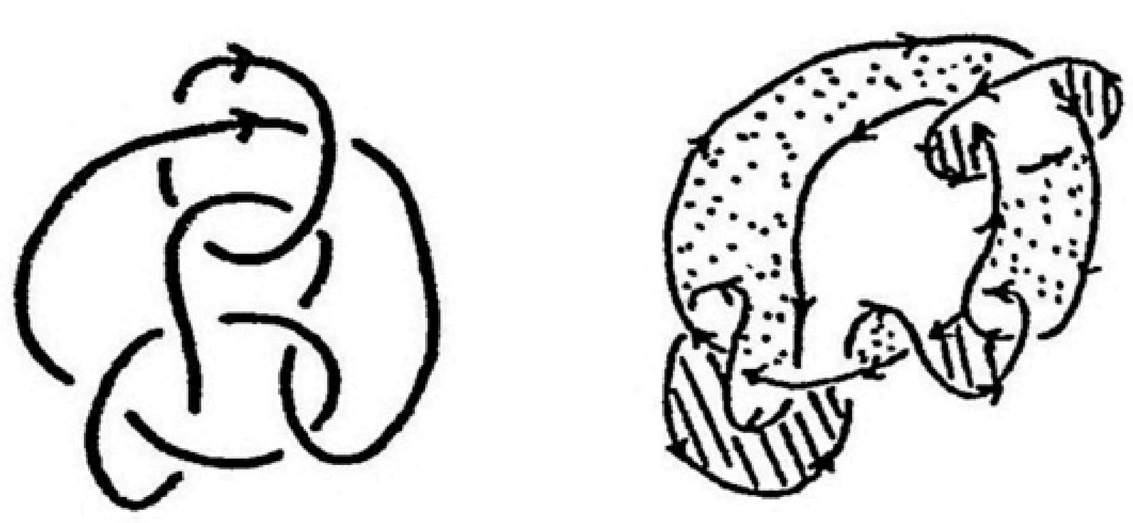
9^2_{54}②



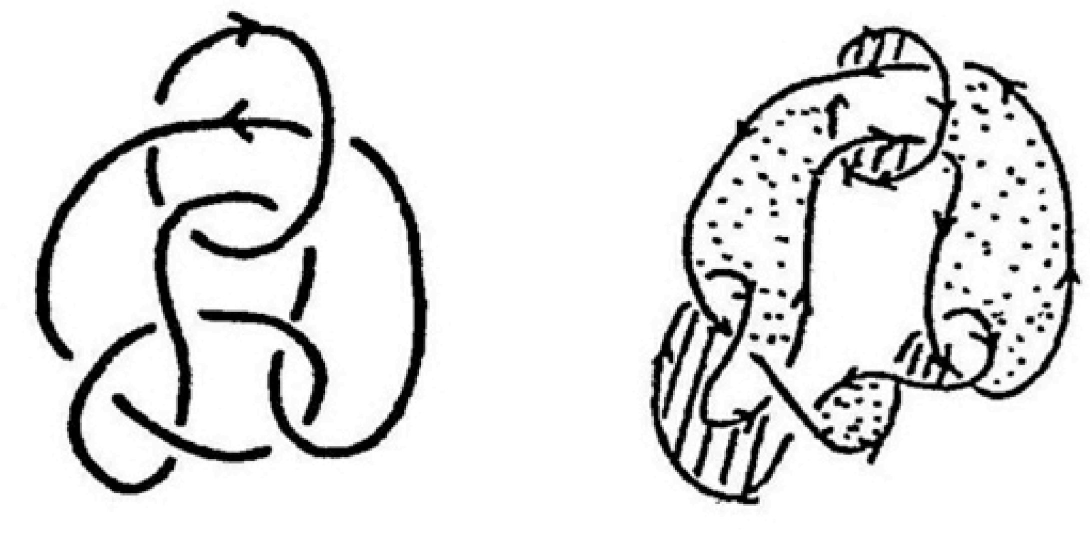
9^2_{55}①



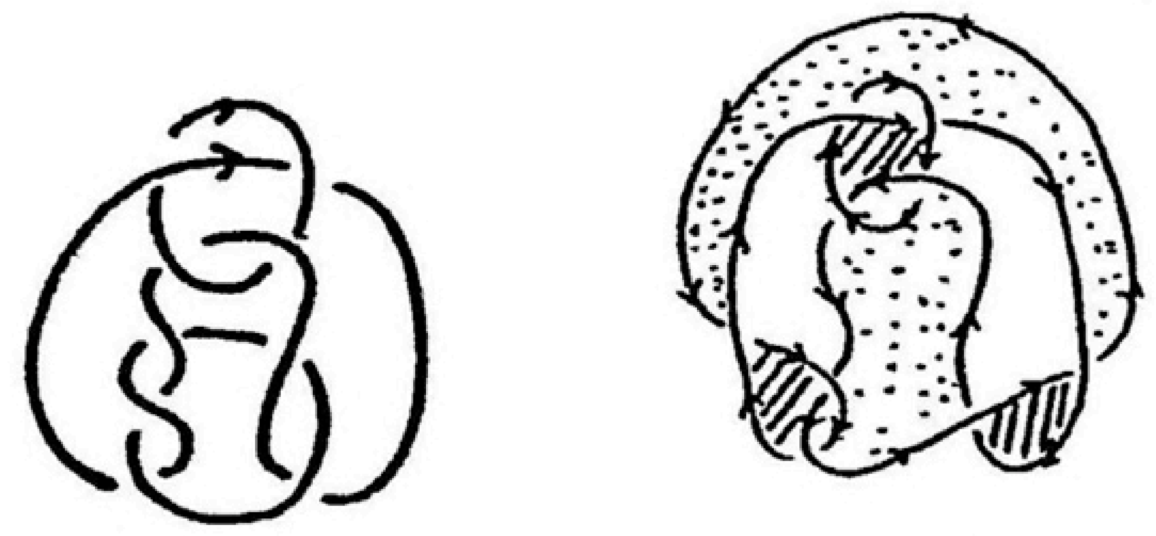
9^2_{55}②



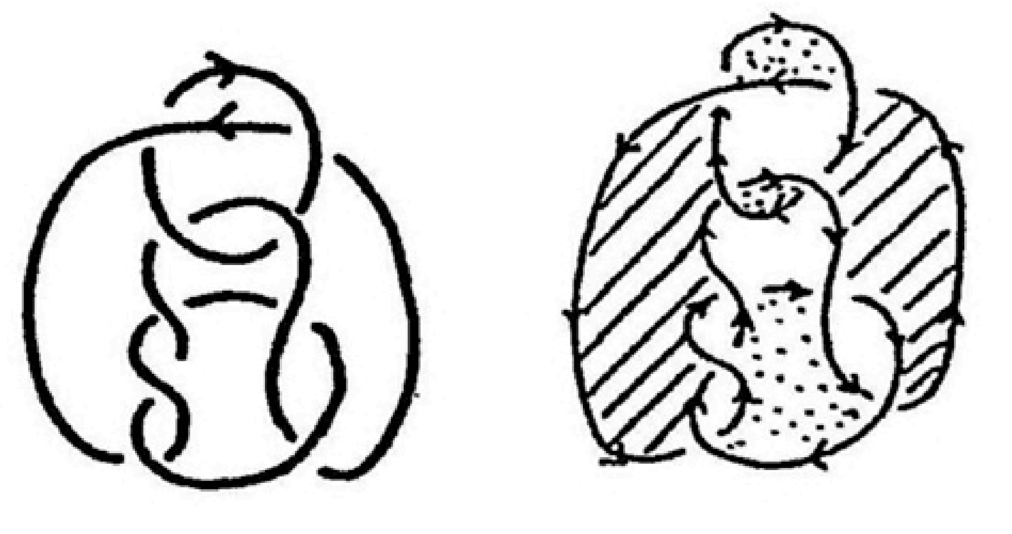
9^2_{56}①



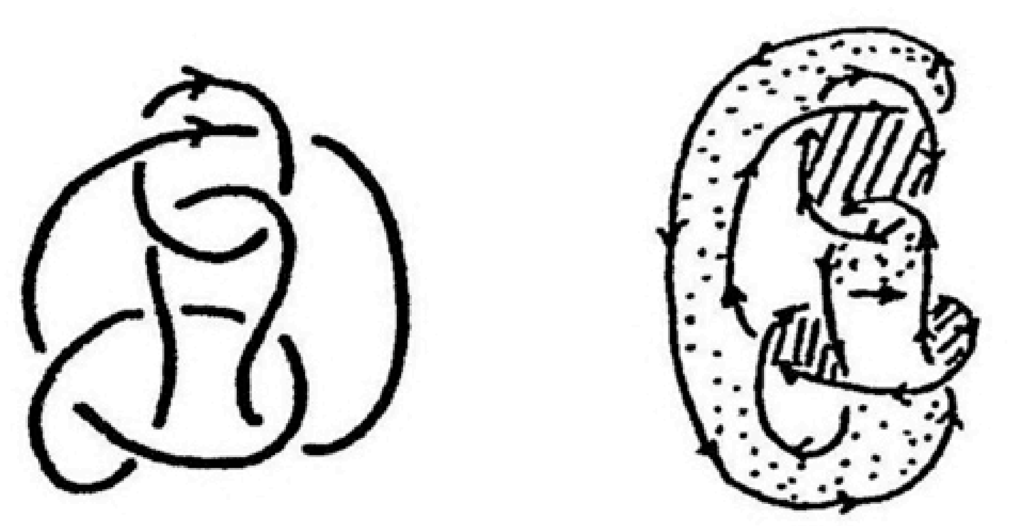
9^2_{56}②



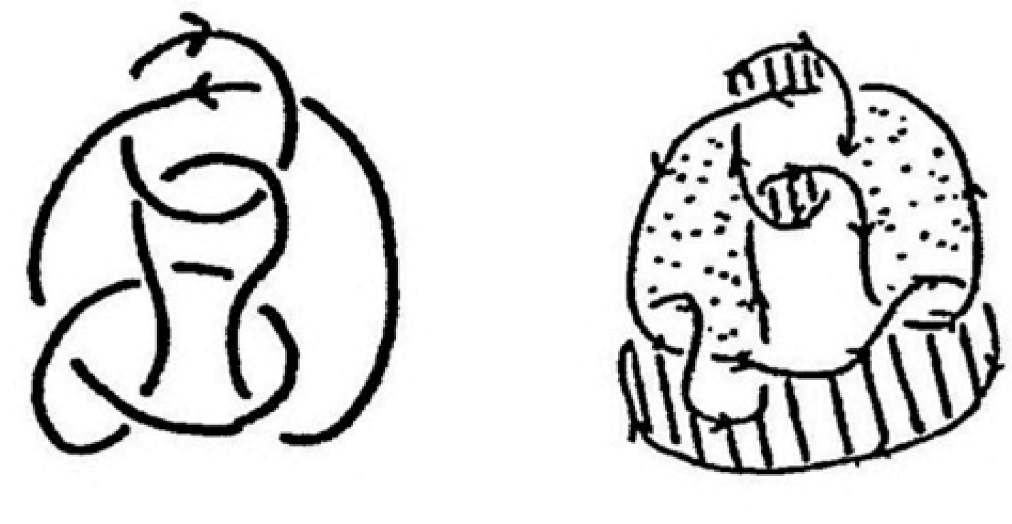
9^2_{57}①



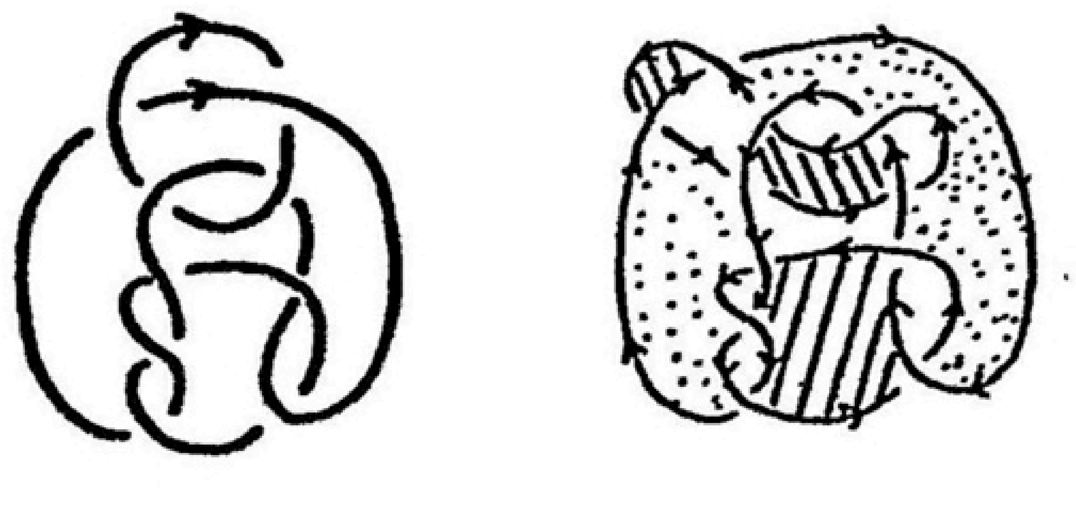
9^2_{57}②



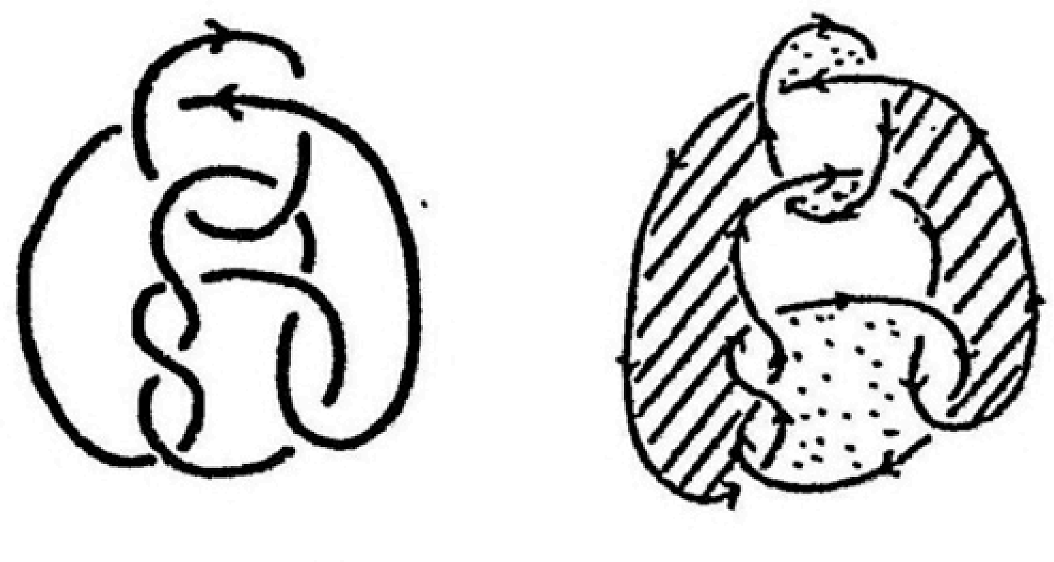
9^2_{58}①



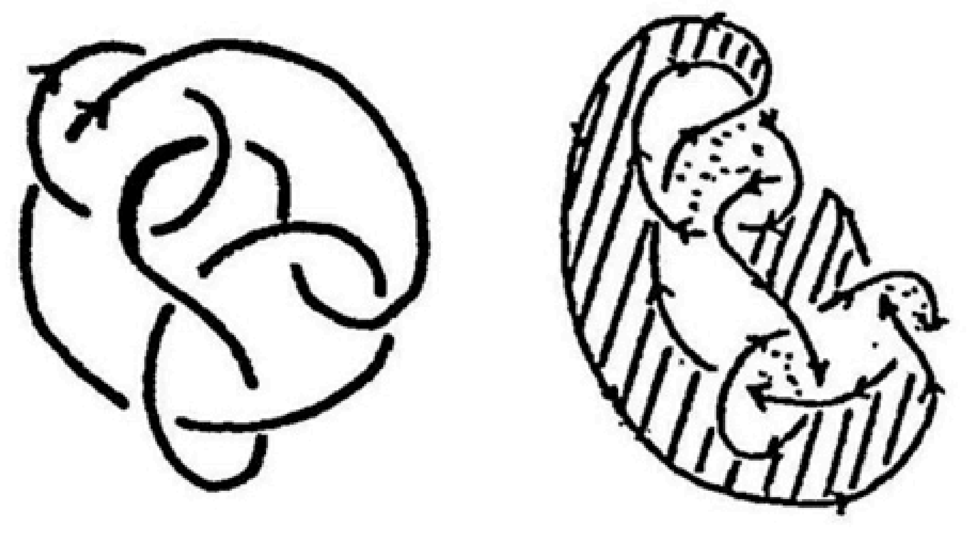
9^2_{58}②



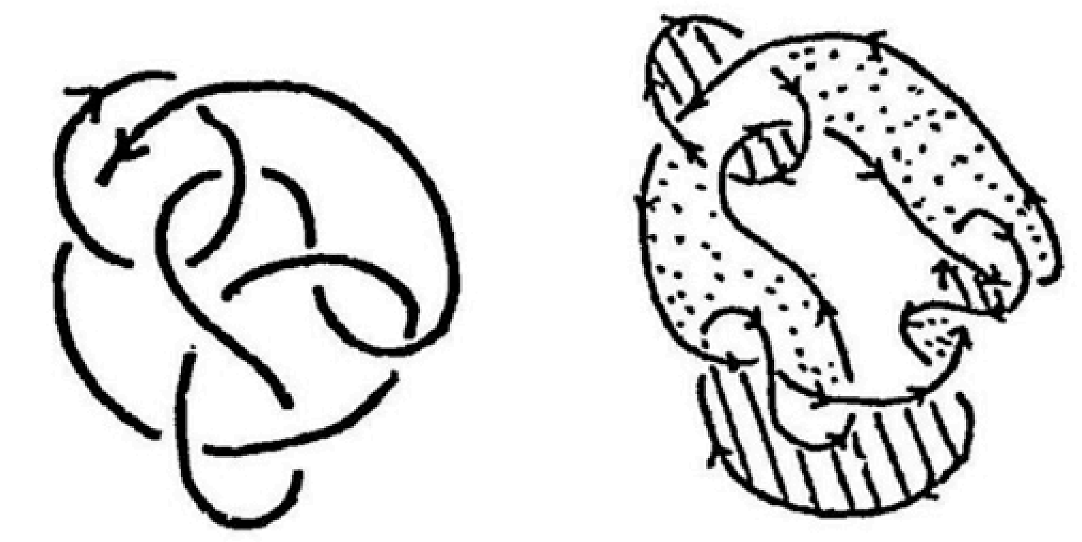
9^2_{59}①



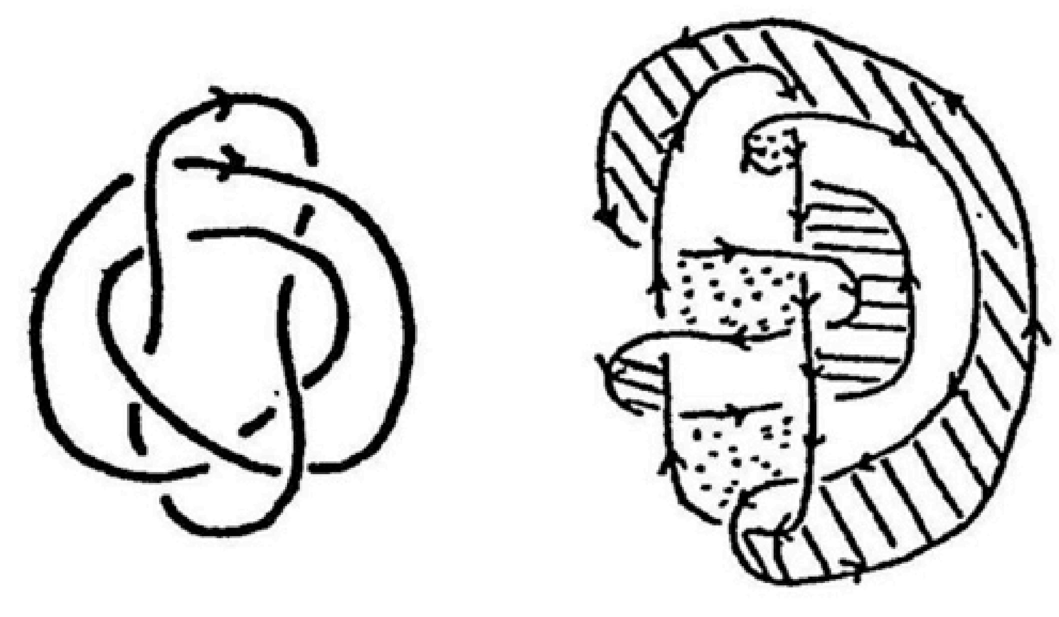
9^2_{59}②



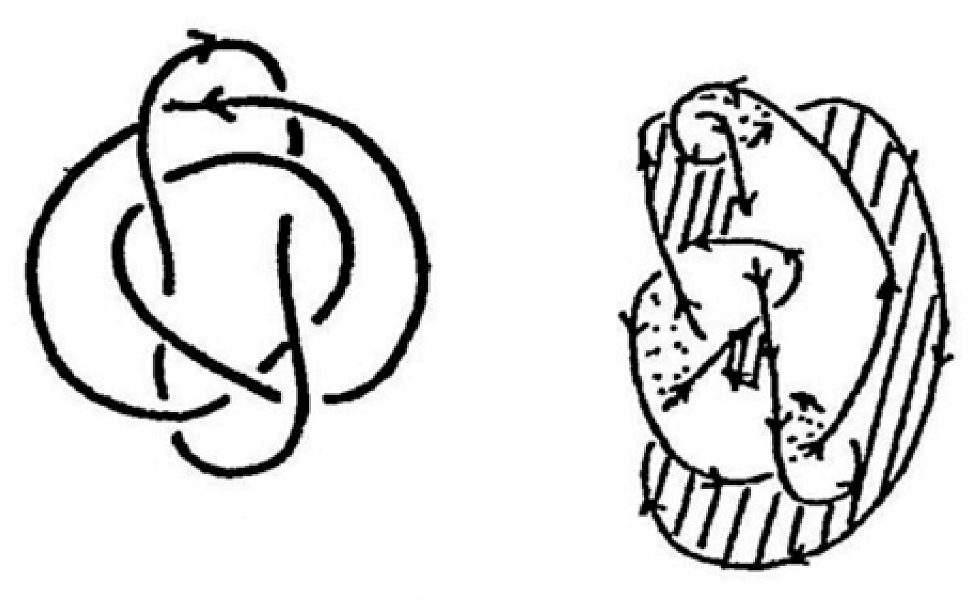
9^2_{60}①



9^2_{60}②

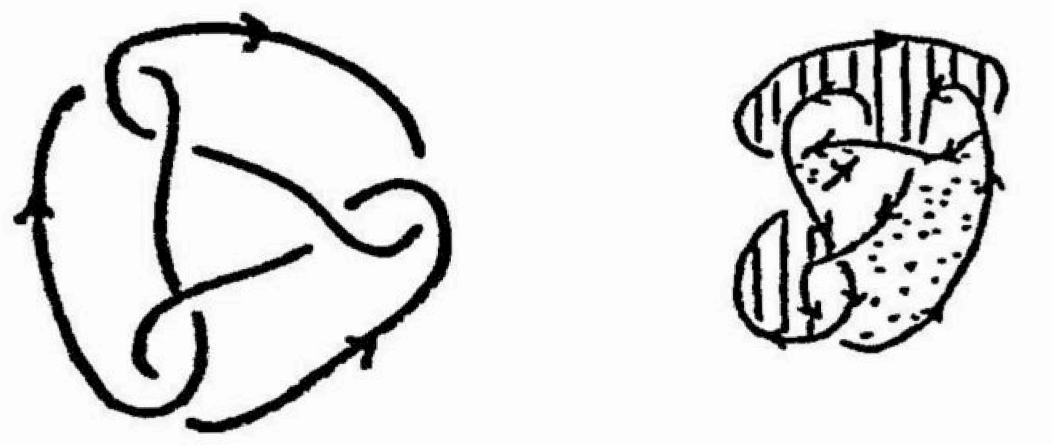


9^2_{61}①

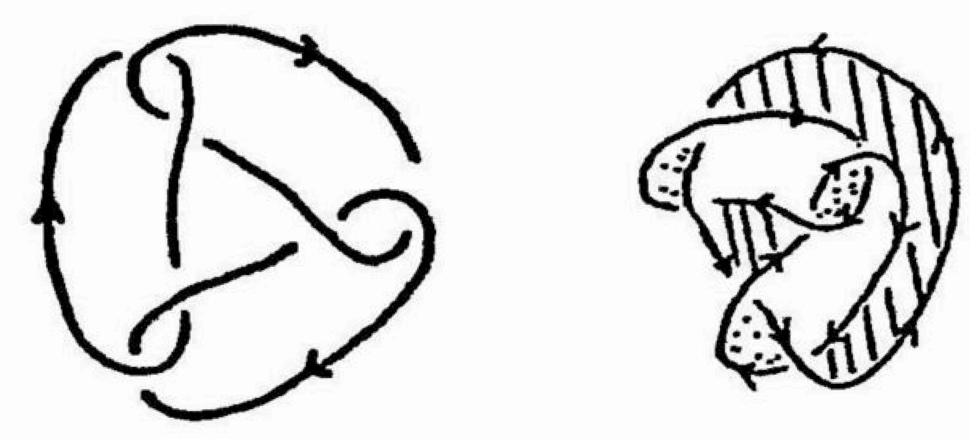


9^2_{61}②

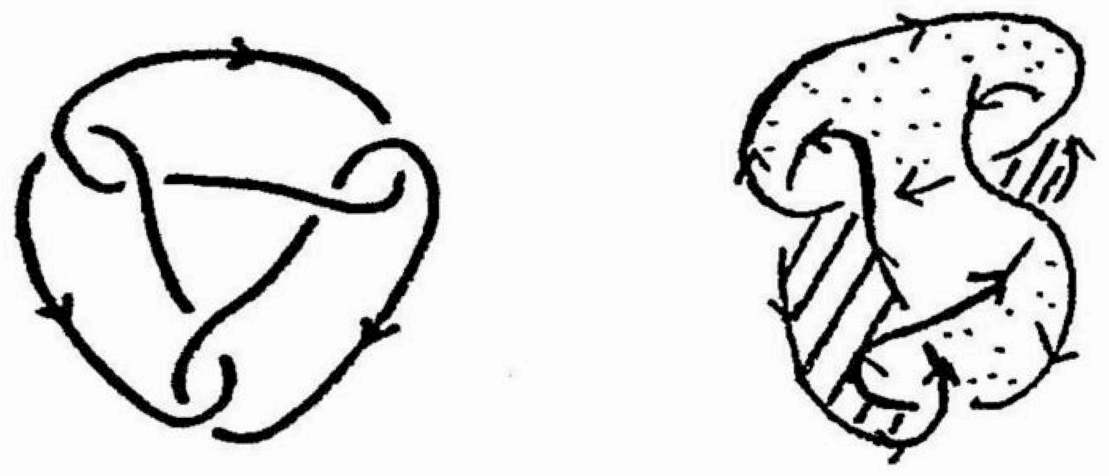
A.2 3-component prime links



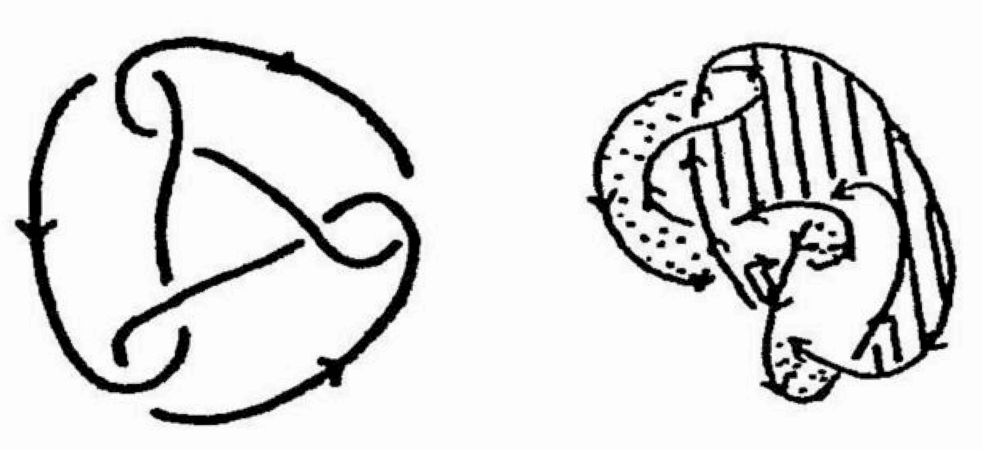
$6^3_1$①



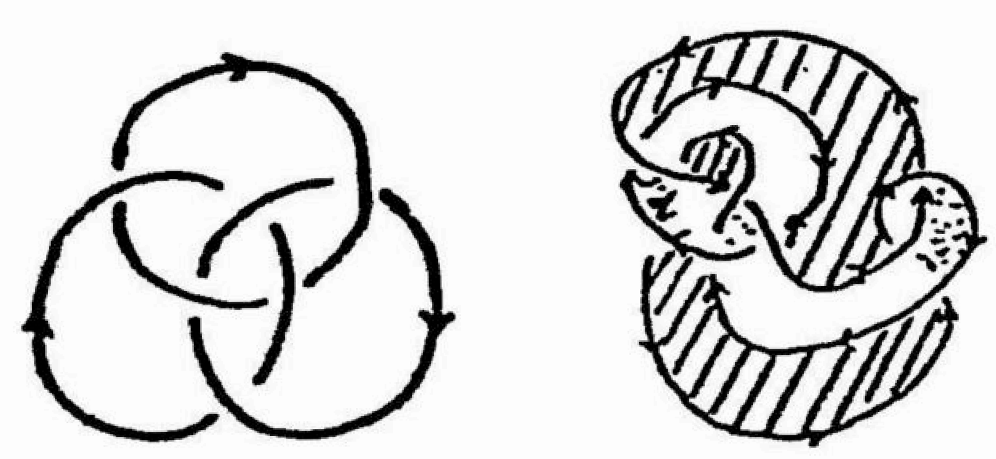
$6^3_1$②



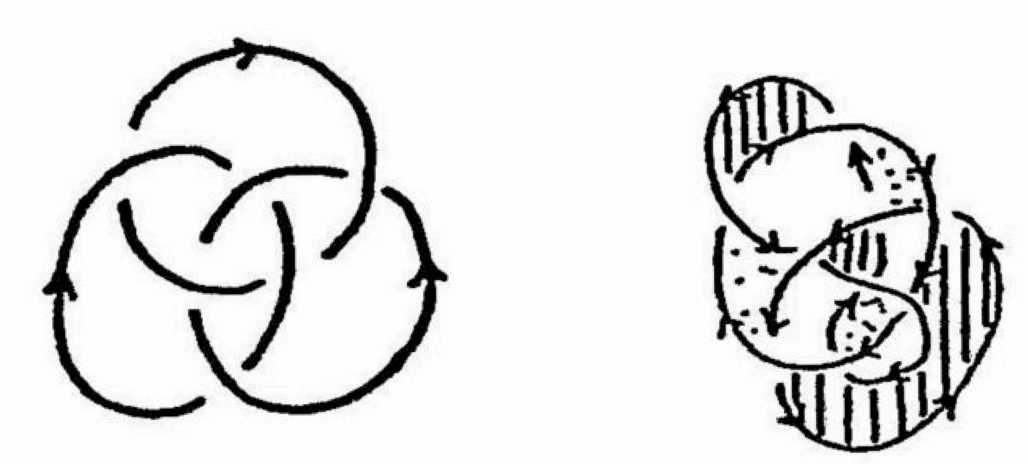
$6^3_1$③



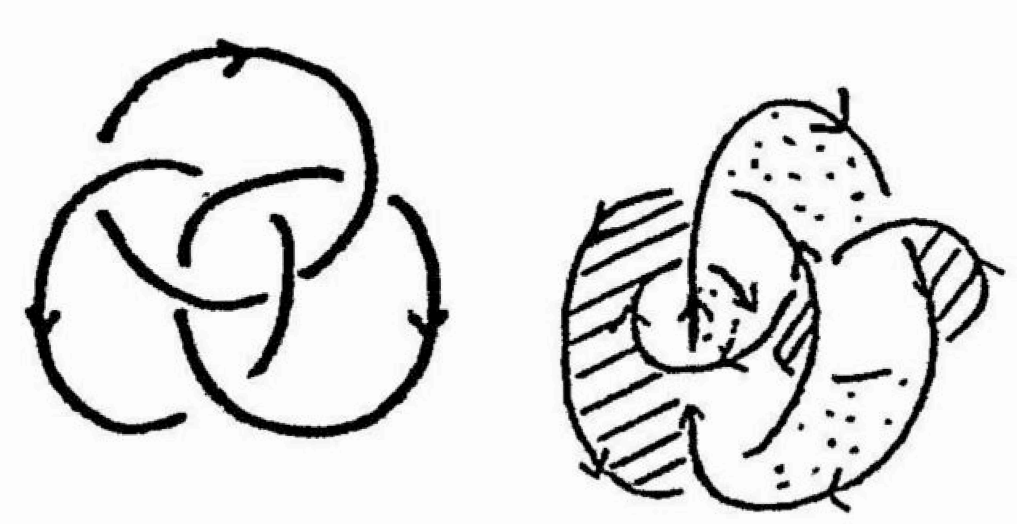
$6^3_1$④



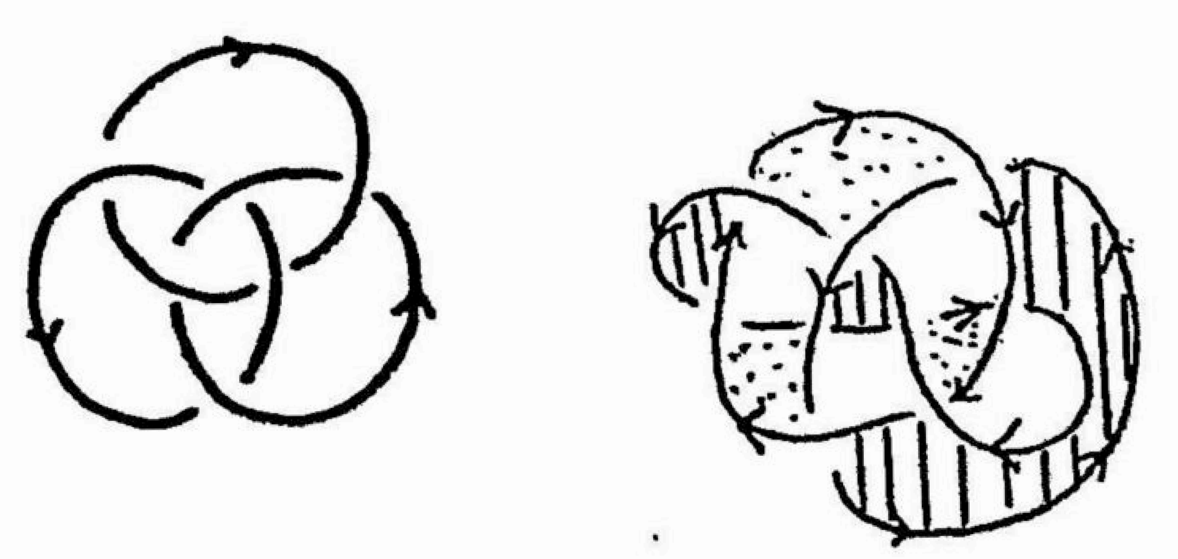
$6^3_2$①



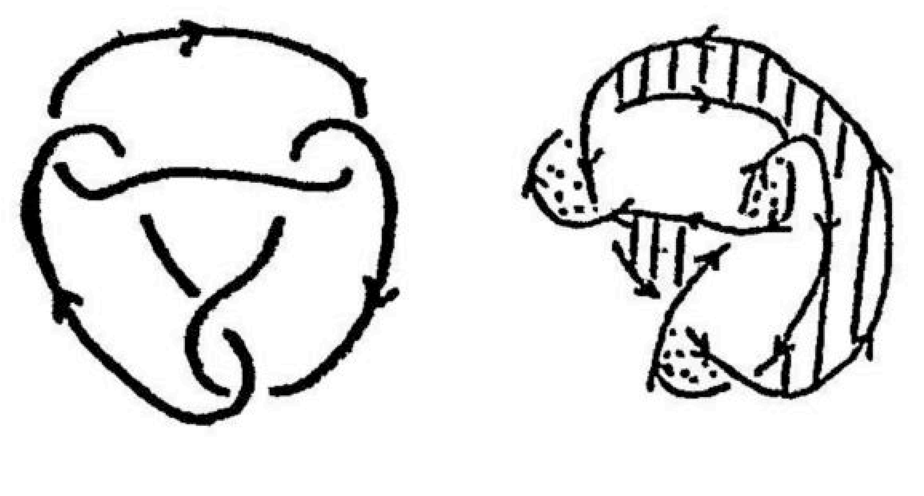
$6^3_2$②



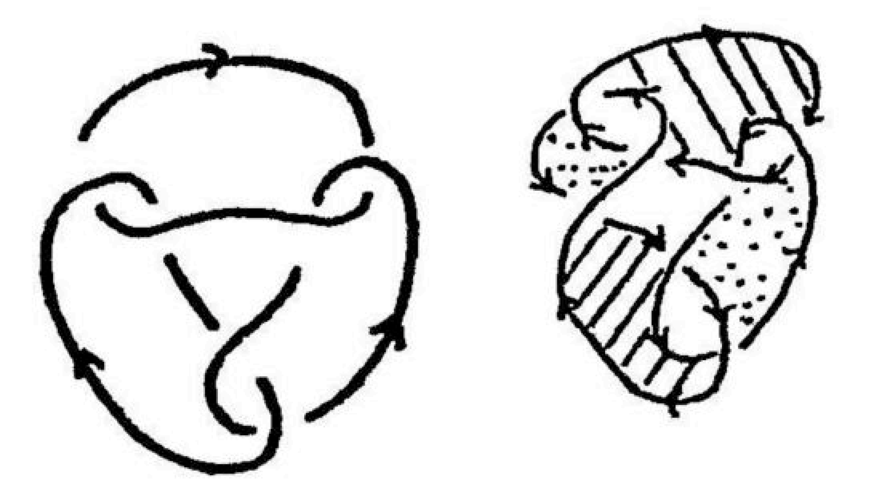
$6^3_2$③



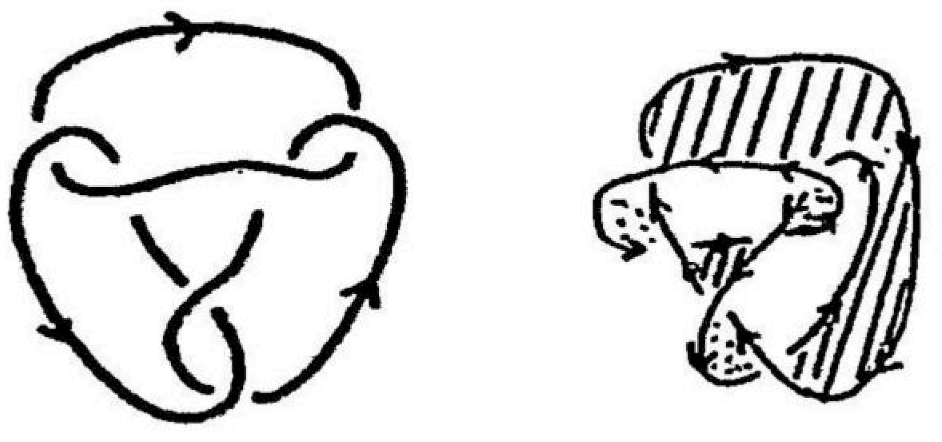
$6^3_2$④



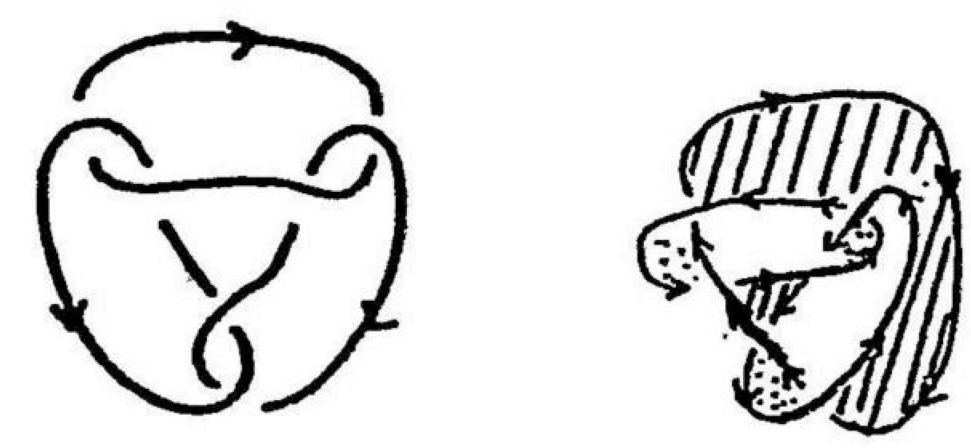
$6^3_3$①



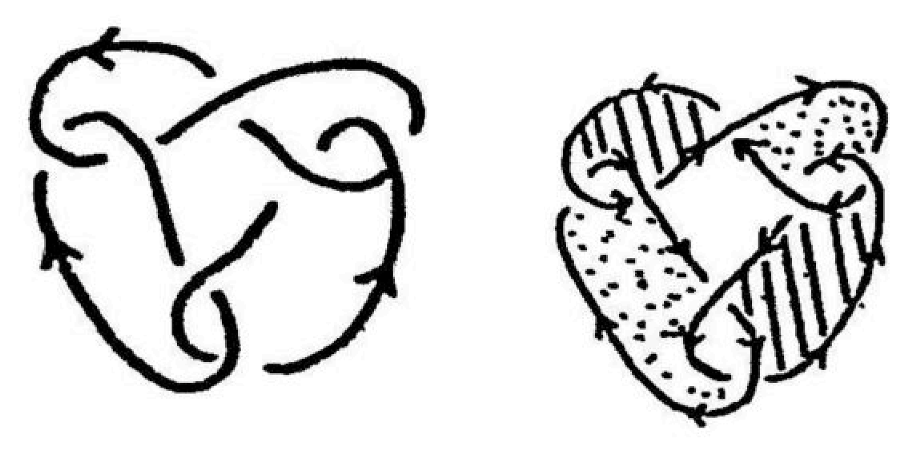
$6^3_3$②



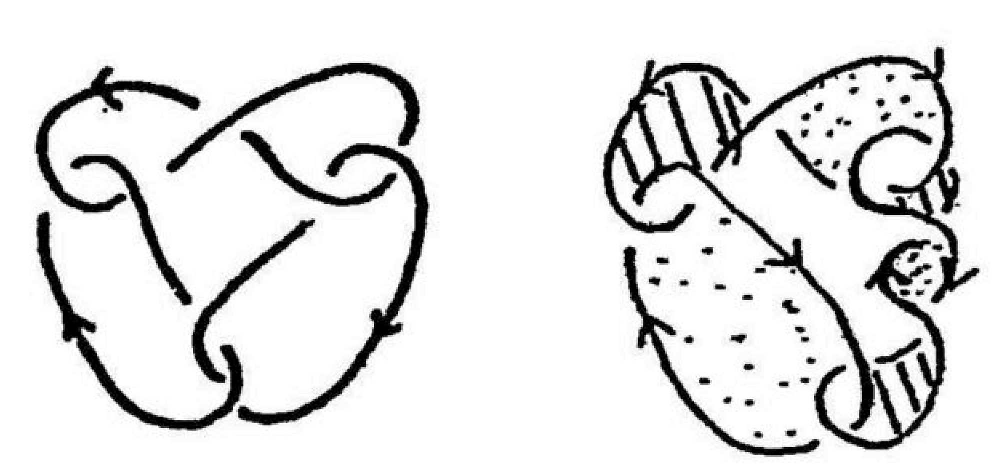
$6^3_3$③



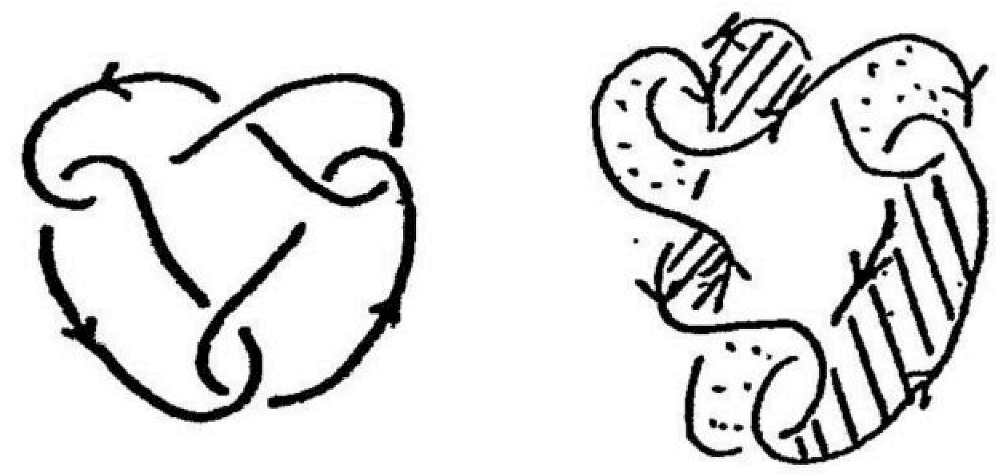
$6^3_3$④



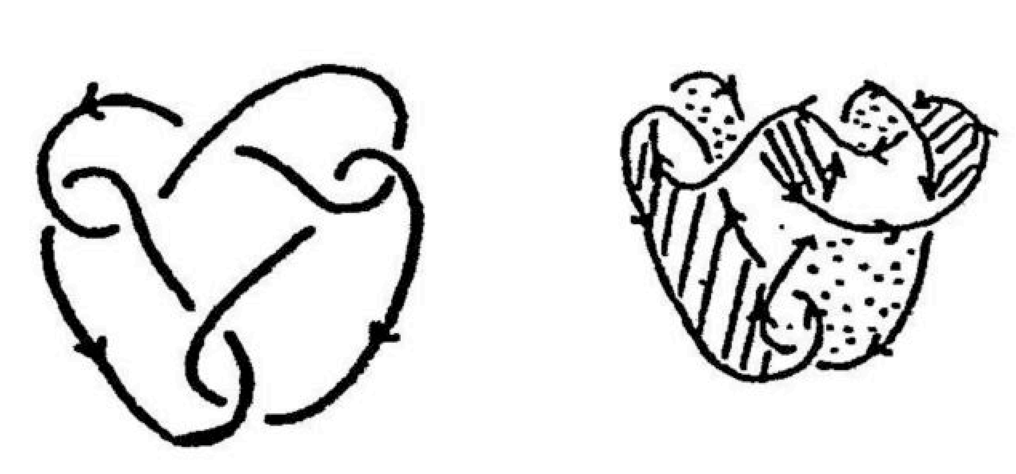
$7^3_1$①



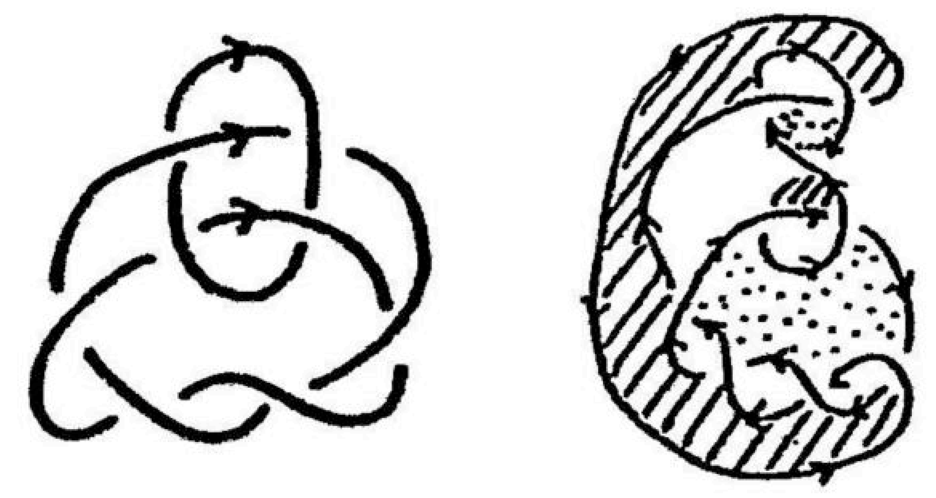
$7^3_1$②



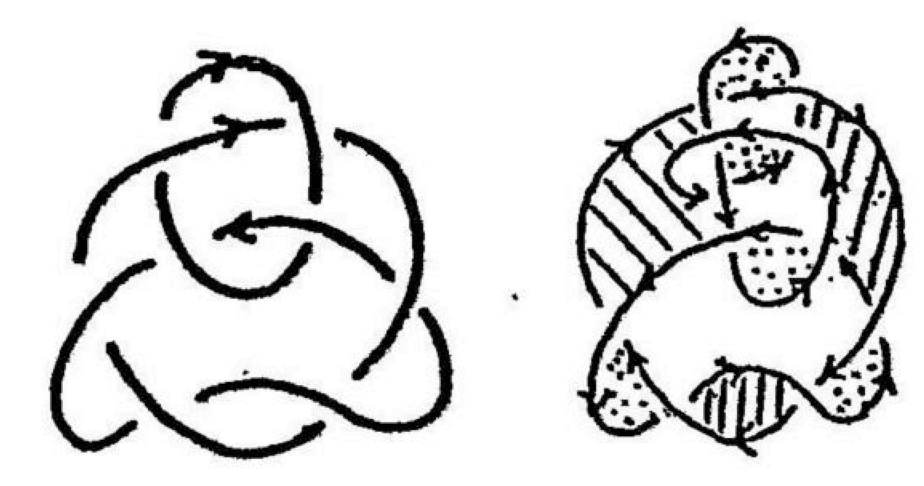
$7^3_1$③



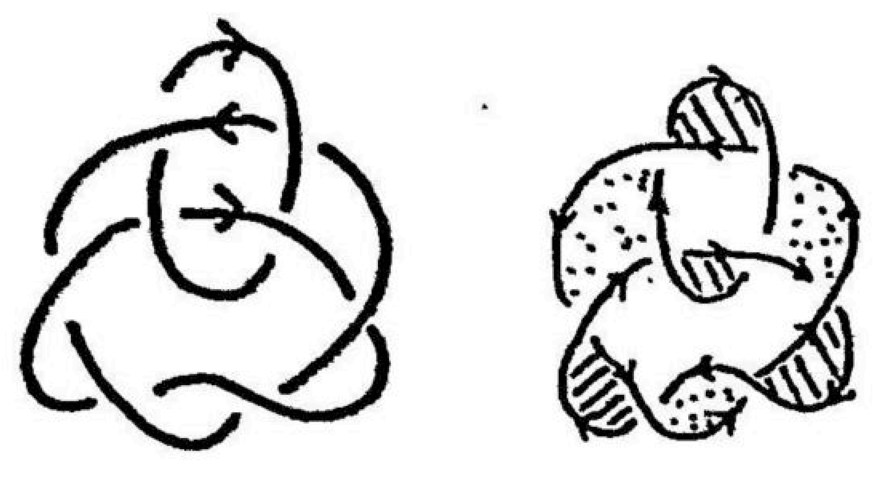
$7^3_1$④



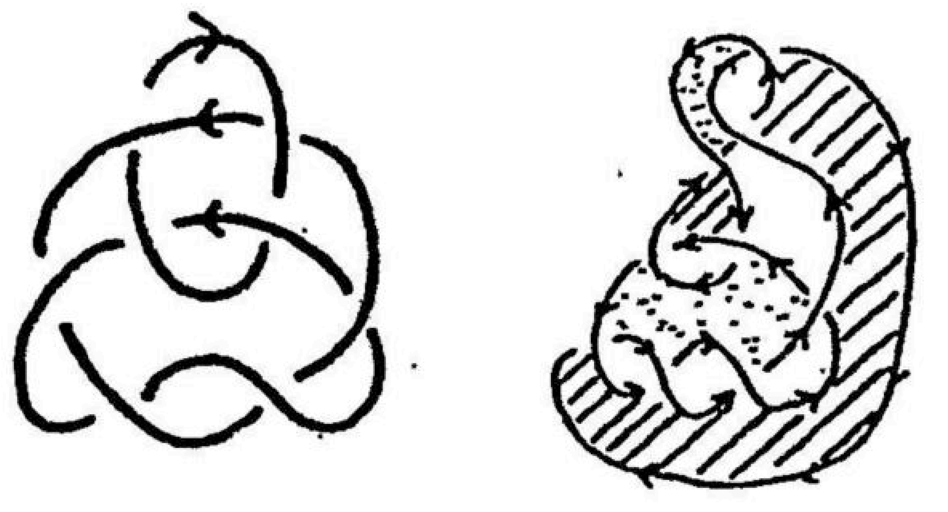
$8^3_1$①



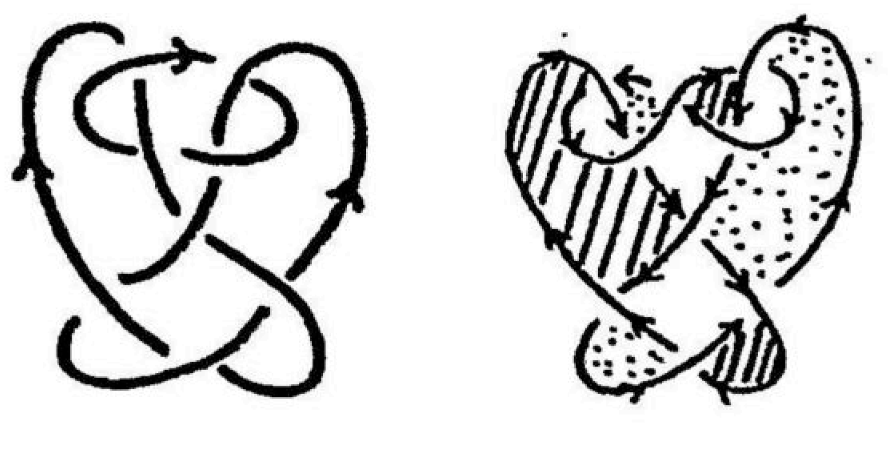
$8^3_1$②



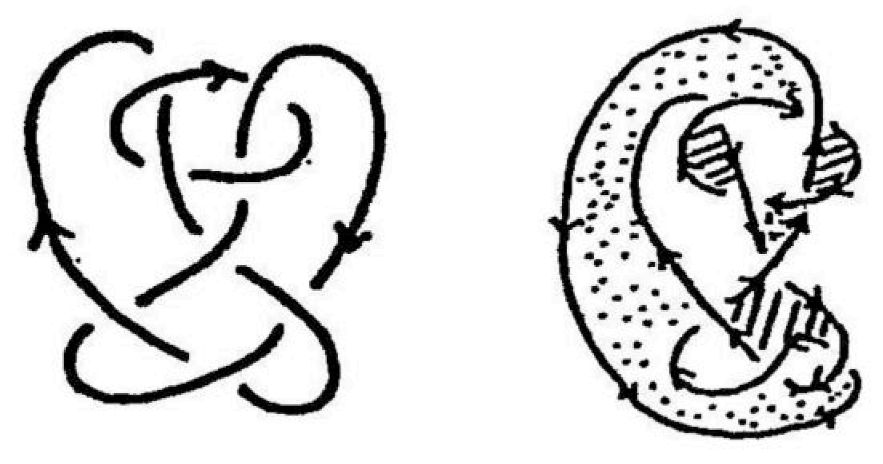
$8^3_1$③



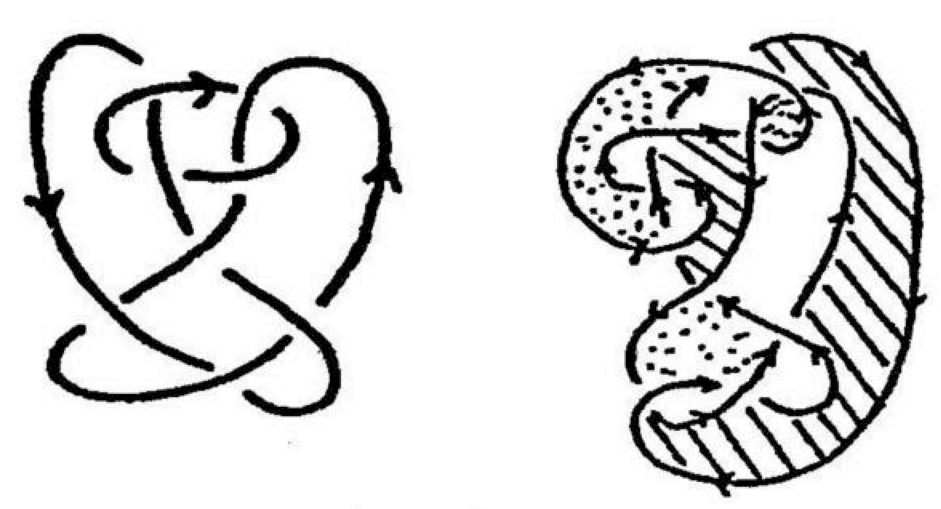
$8^3_1$④



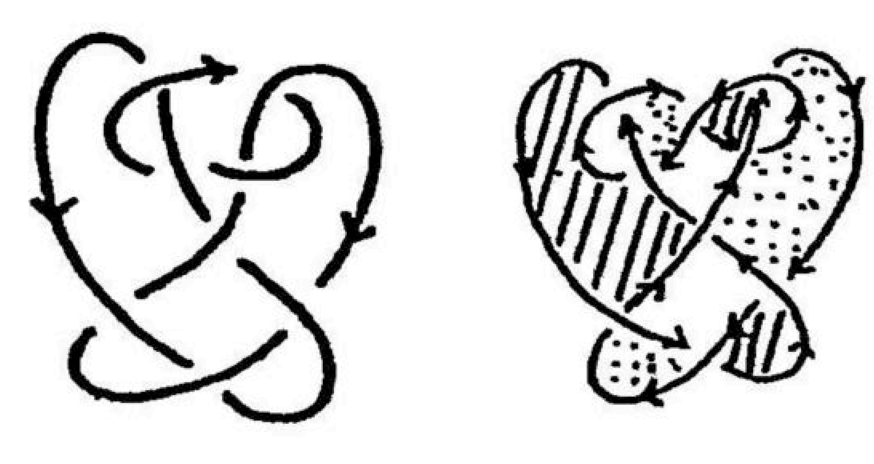
$8^3_2$①



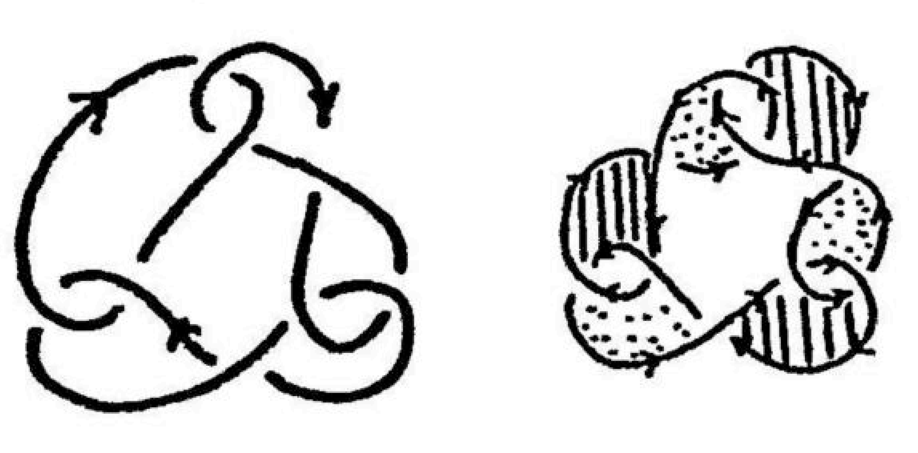
$8^3_2$②



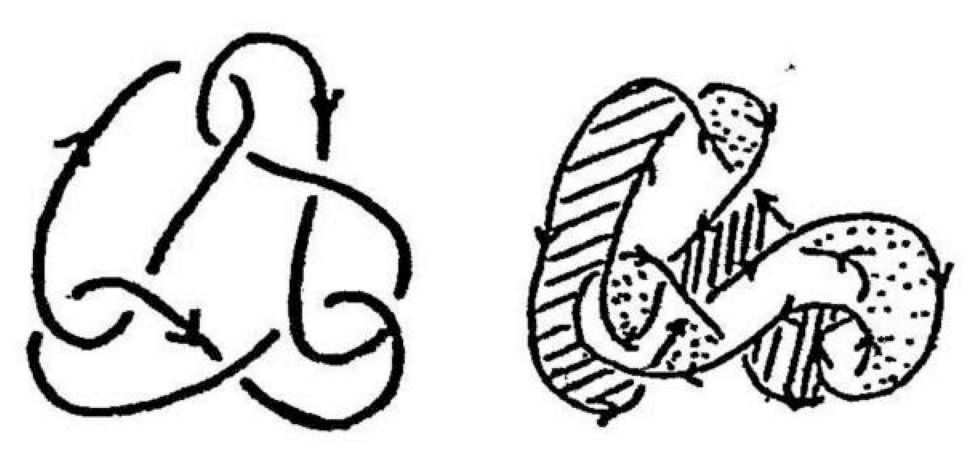
$8^3_2$③



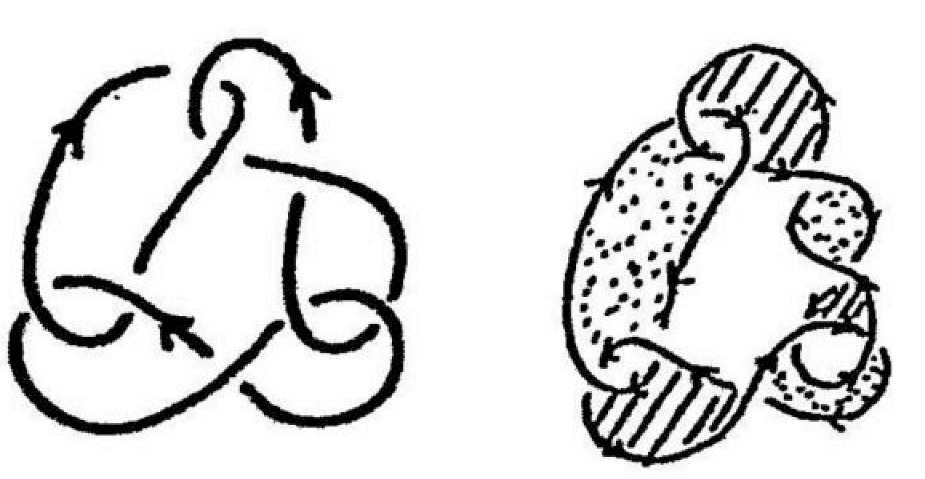
$8^3_2$④



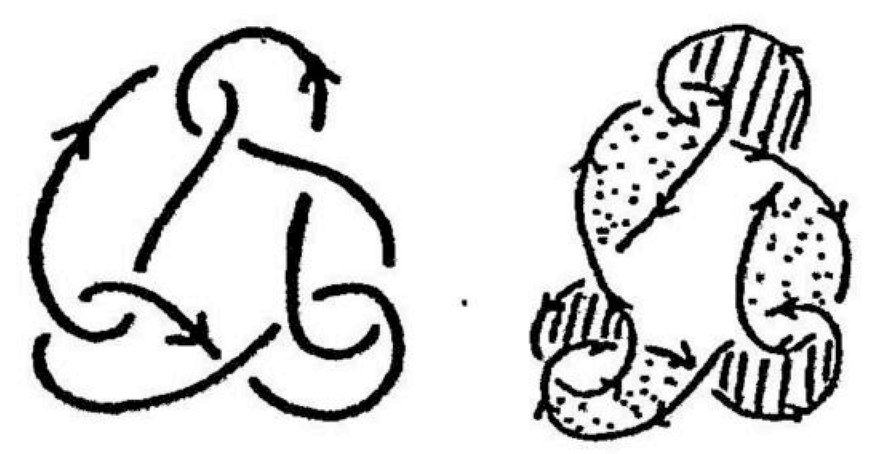
$8^3_3$①



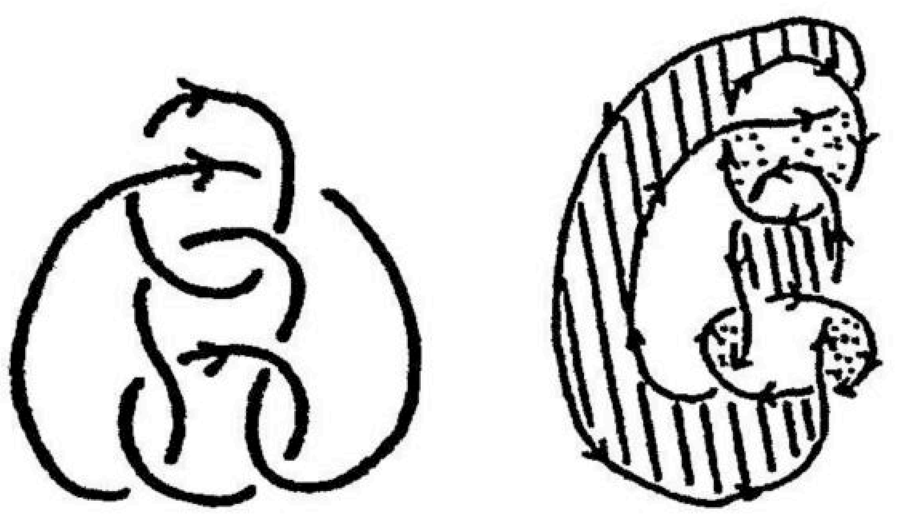
$8^3_3$②



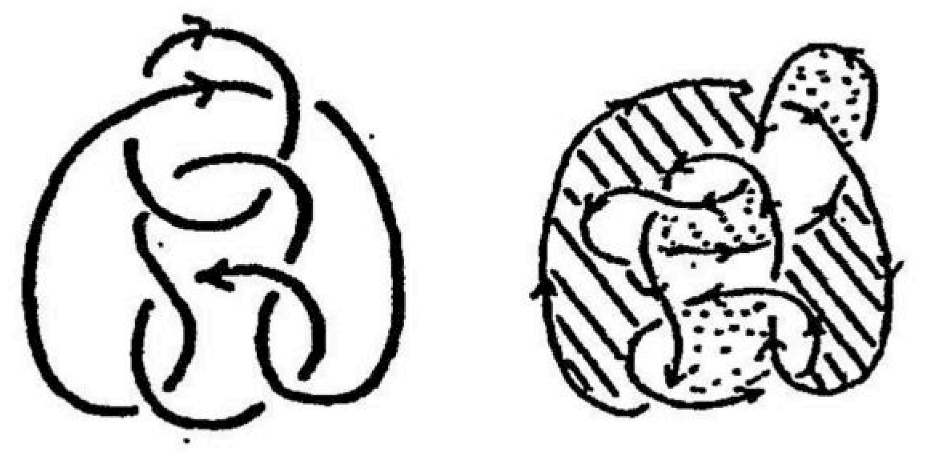
$8^3_3$③



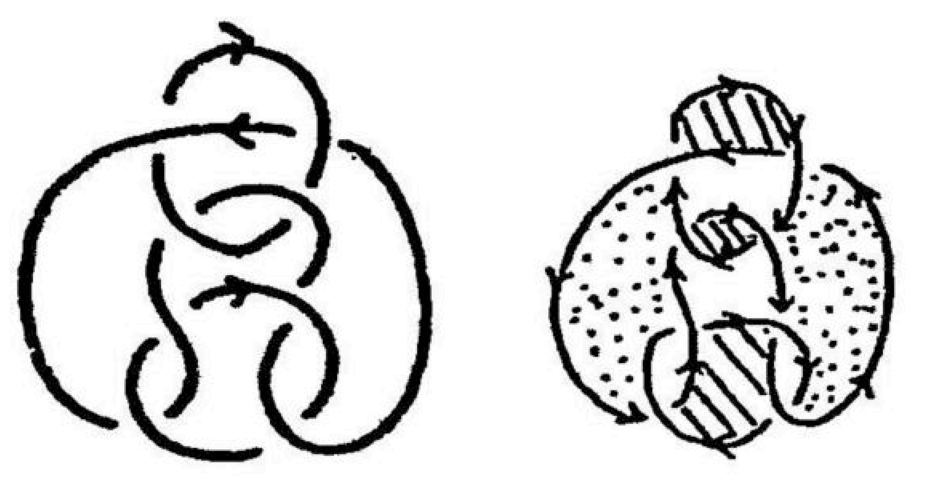
$8^3_3$④



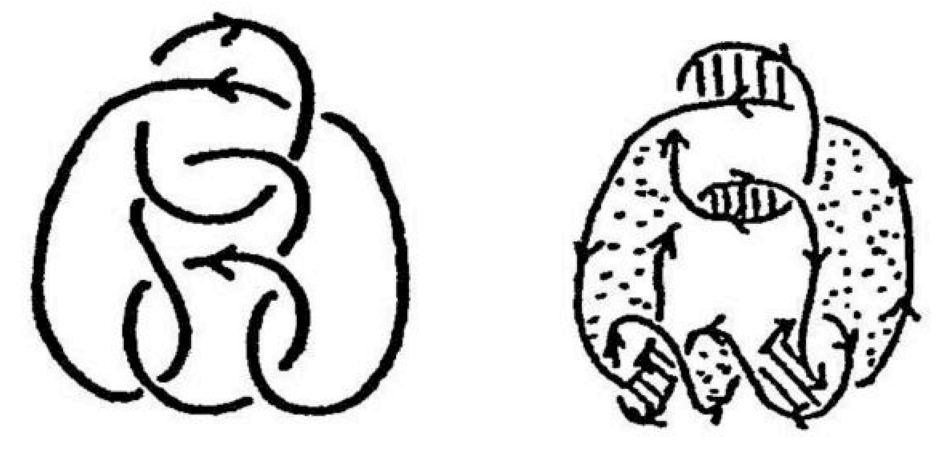
$8^3_4$①



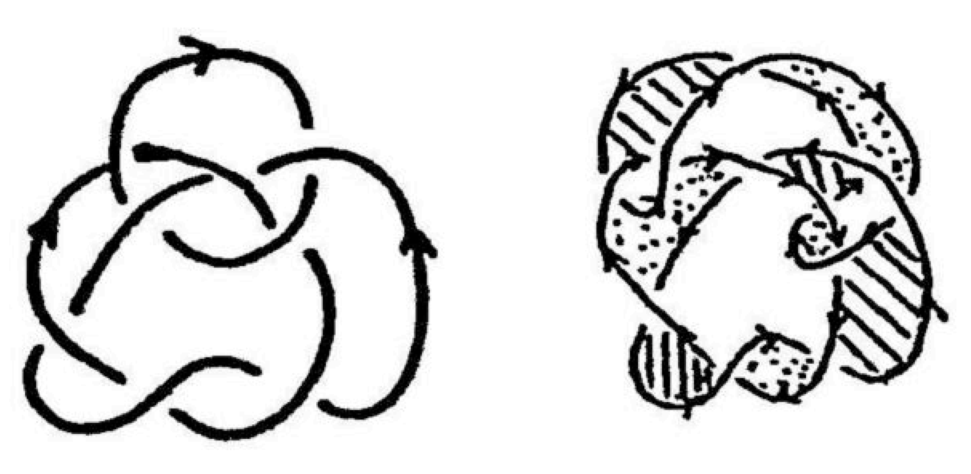
$8^3_4$②



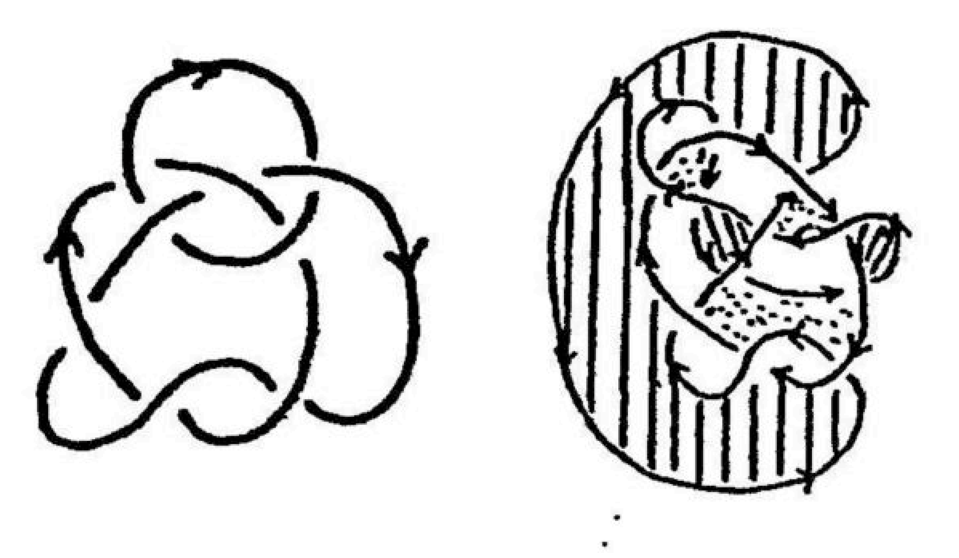
$8^3_4$③



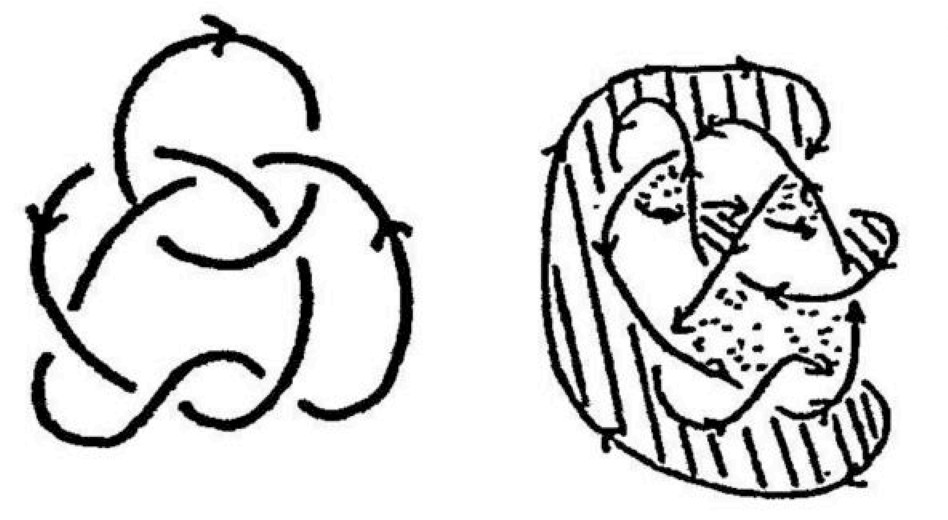
$8^3_4$④



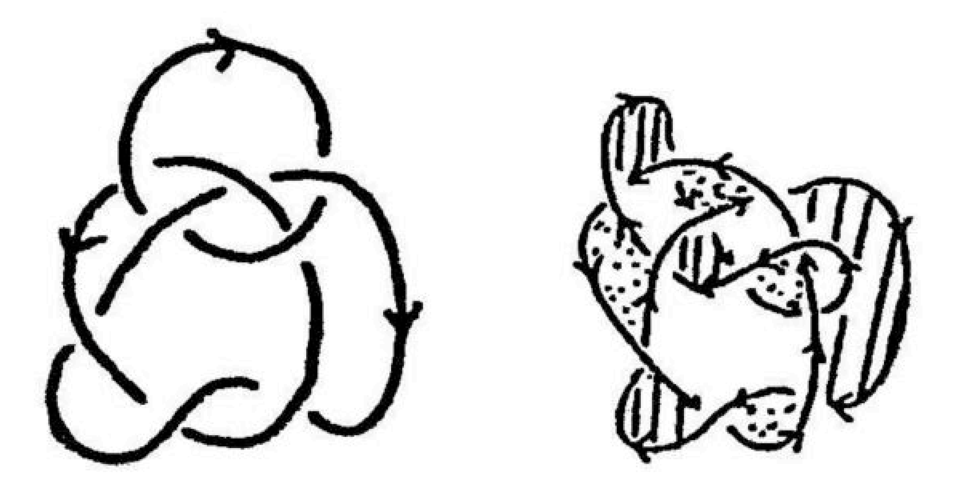
$8^3_5$①



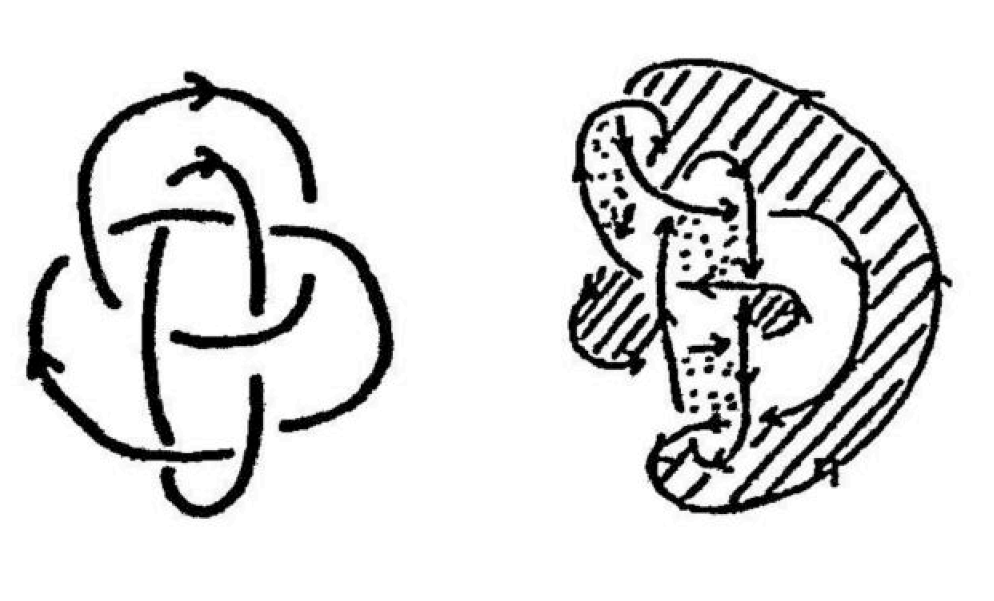
$8^3_5$②



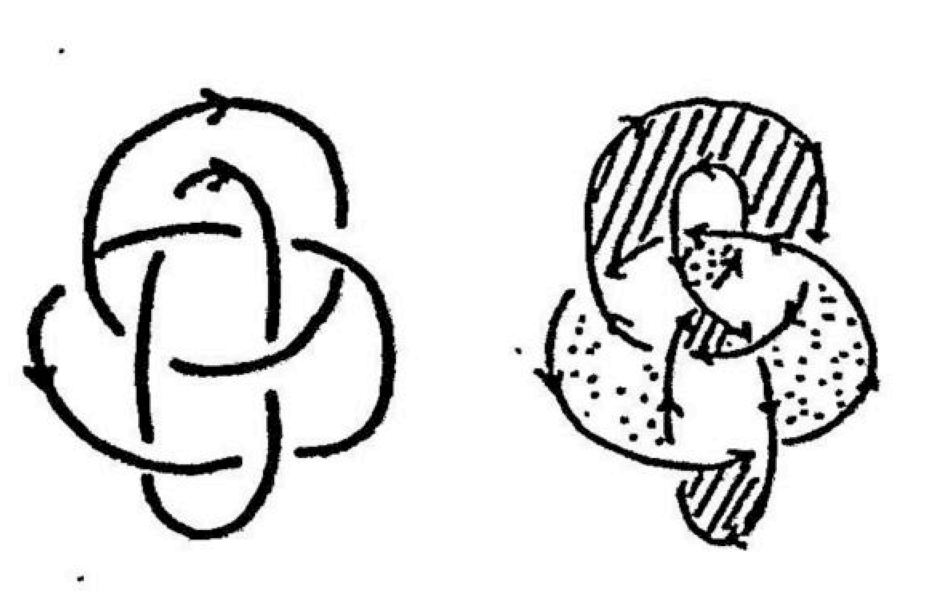
$8^3_5$③



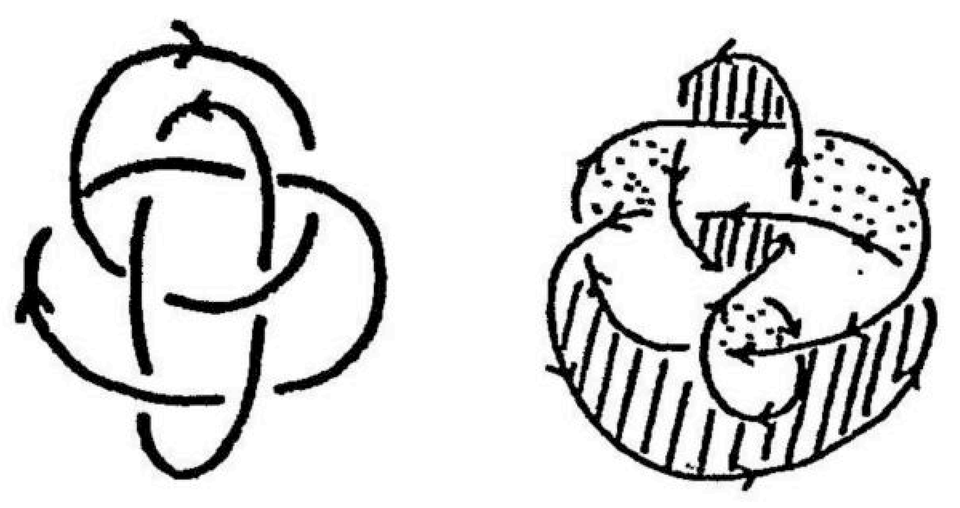
$8^3_5$④



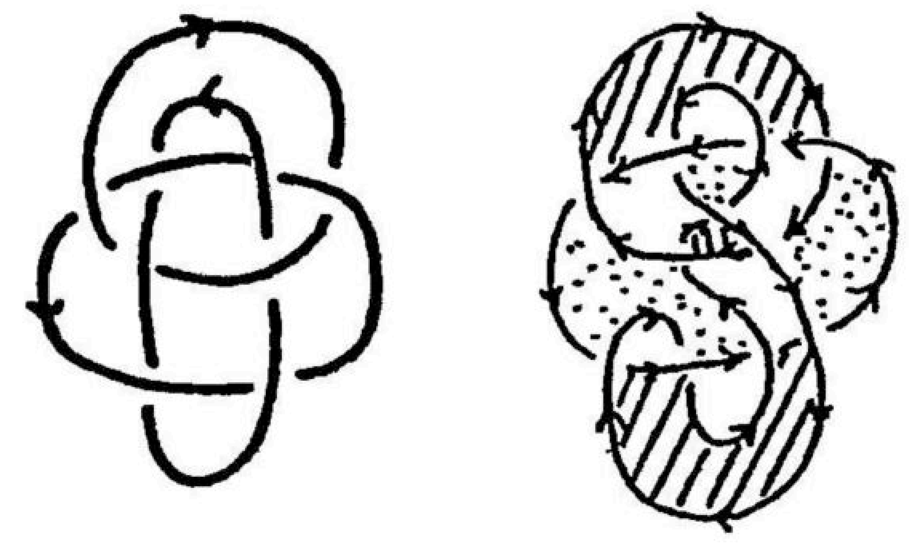
$8^3_6$①



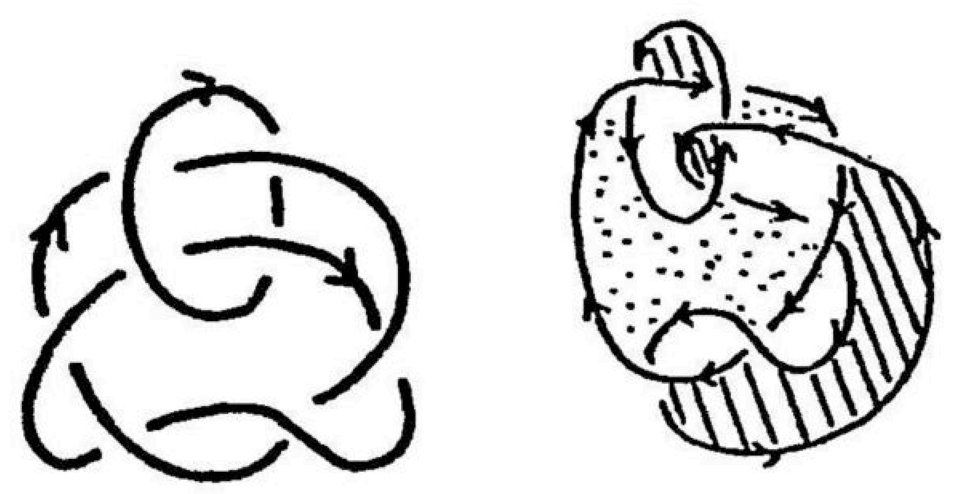
$8^3_6$②



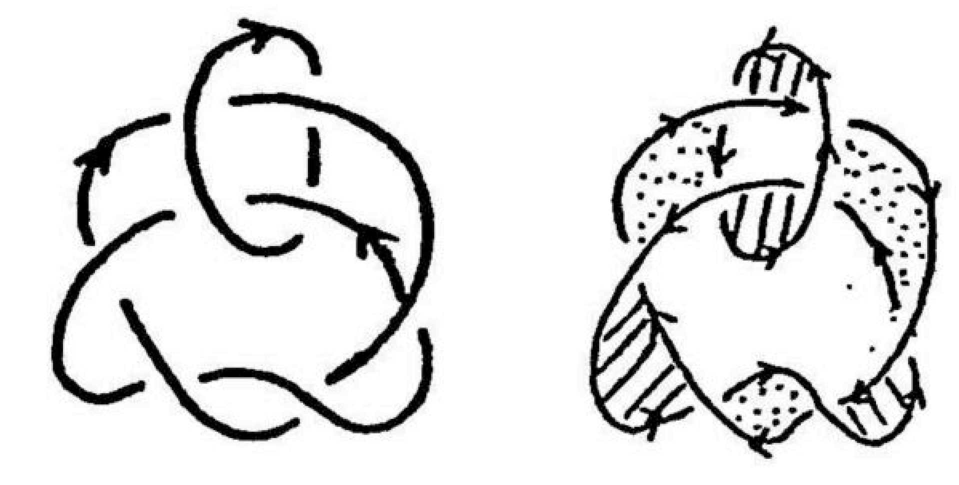
$8^3_6$③



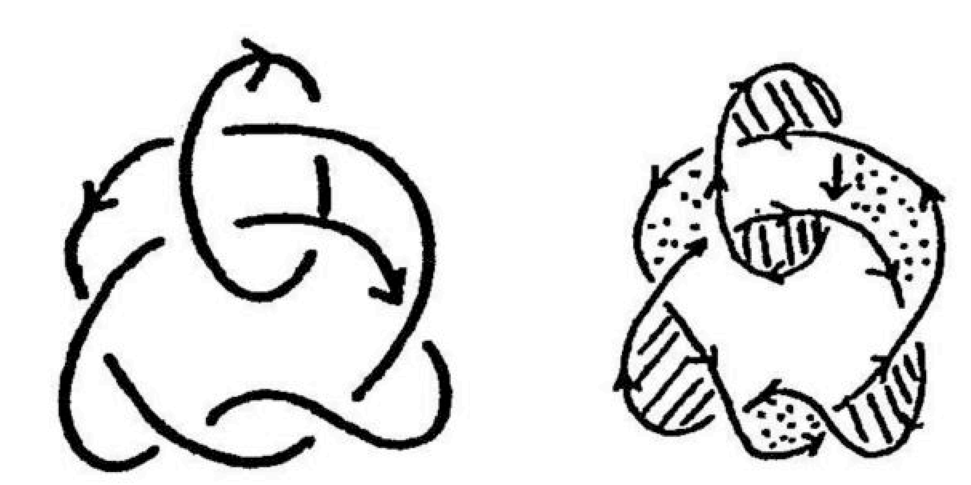
$8^3_6$④



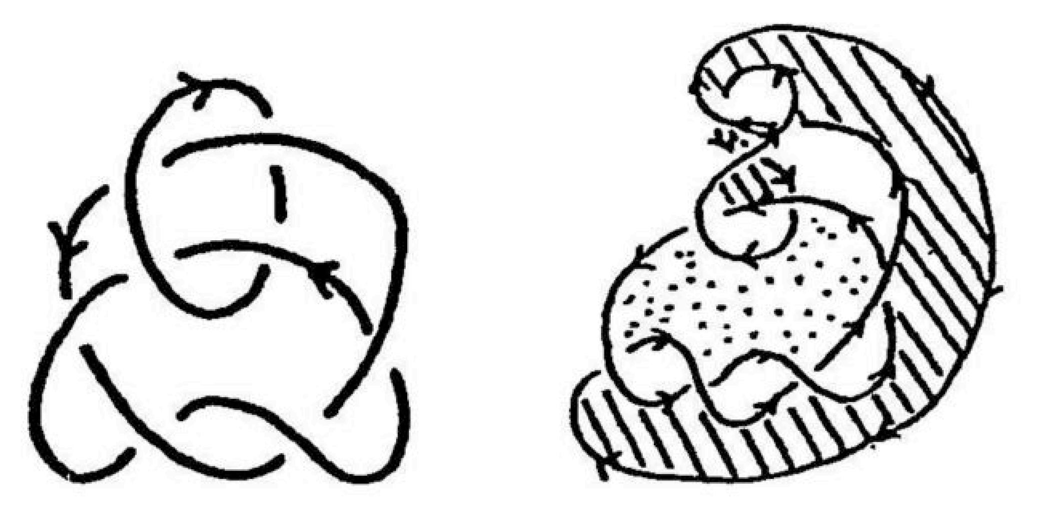
$8^3_7$①



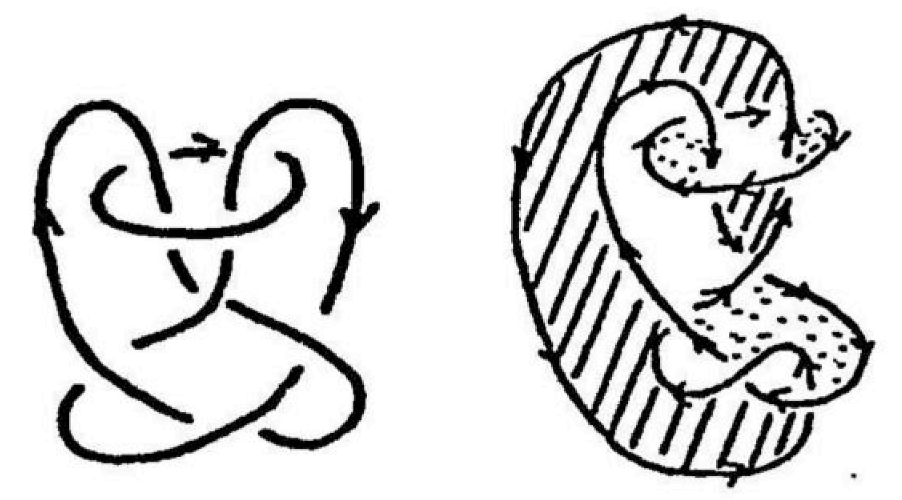
$8^3_7$②



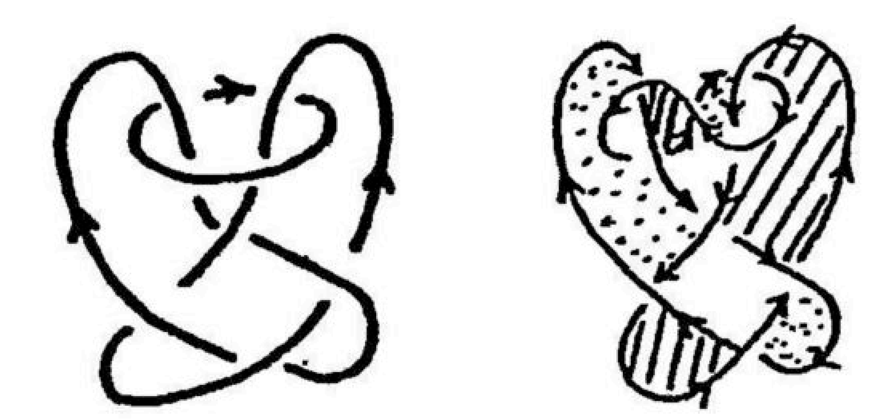
$8^3_7$③



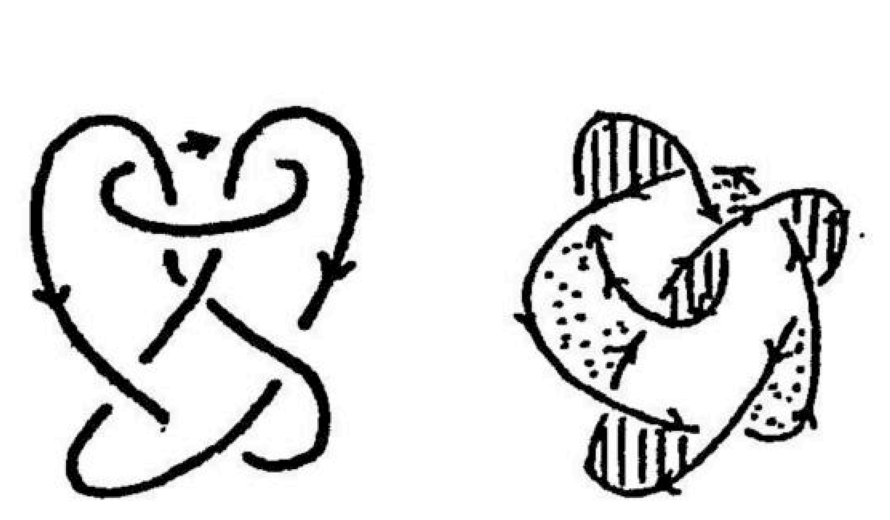
$8^3_7$④



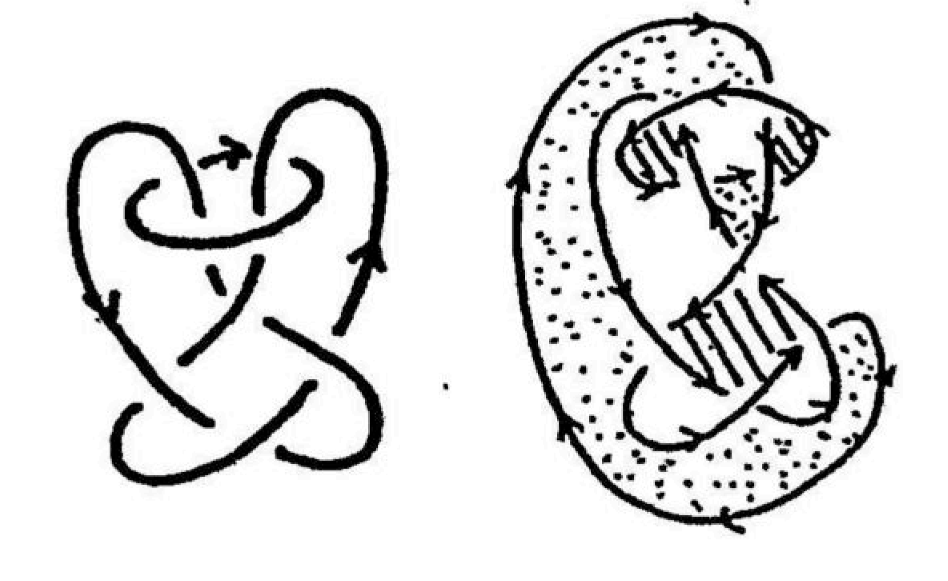
$8^3_8$①



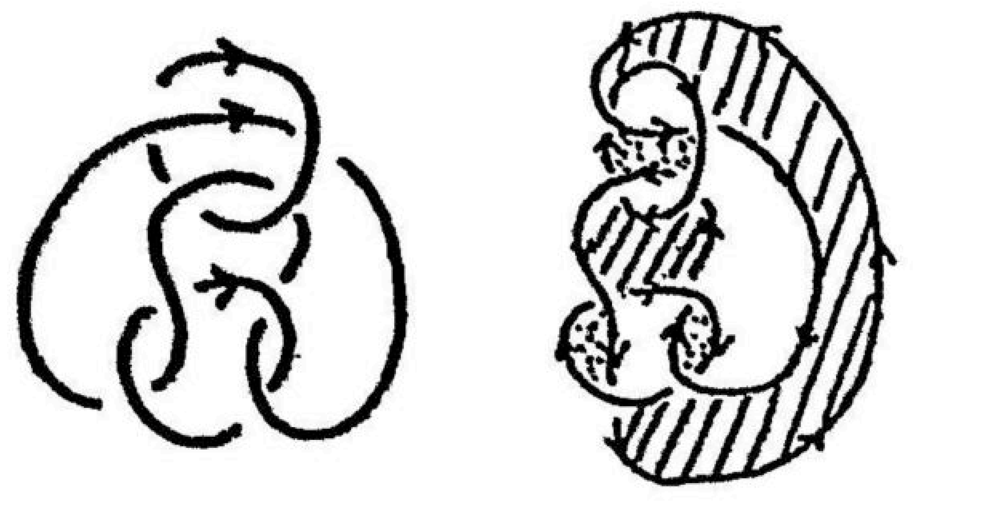
$8^3_8$②



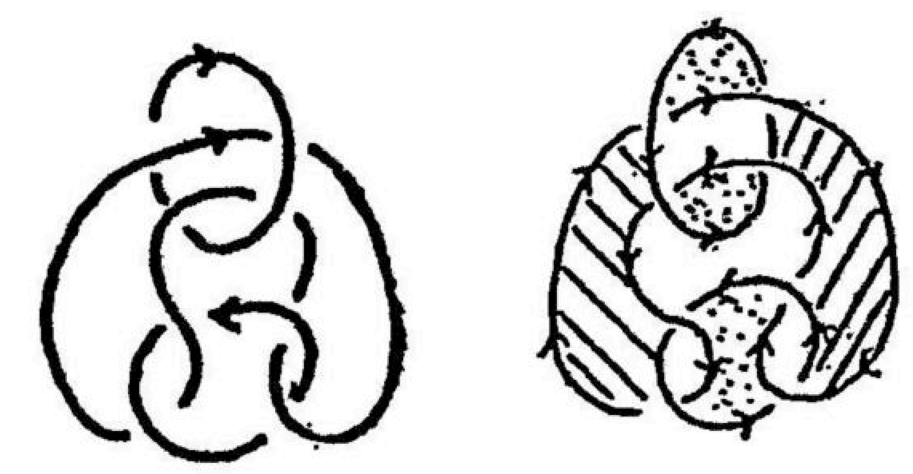
$8^3_8$③



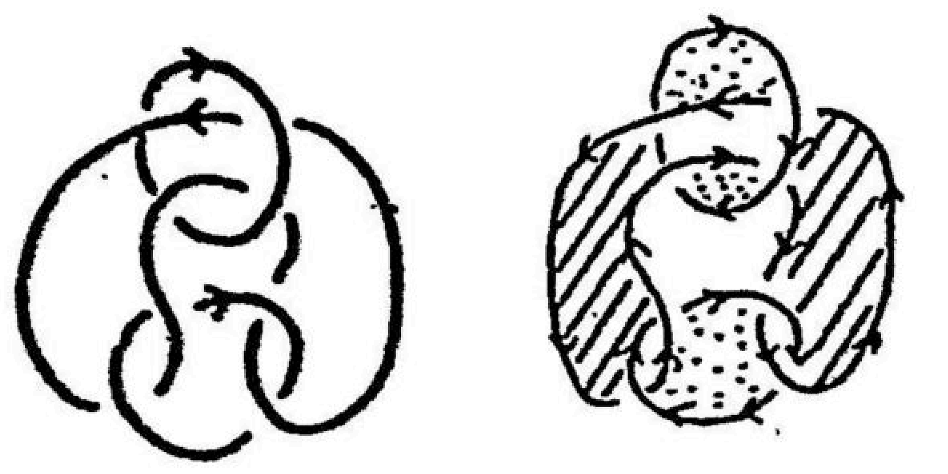
$8^3_8$④



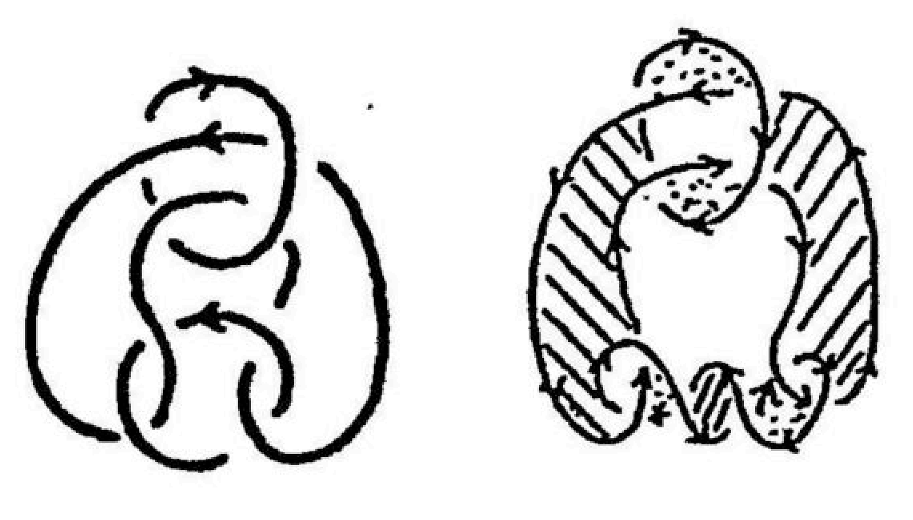
$8^3_9$①



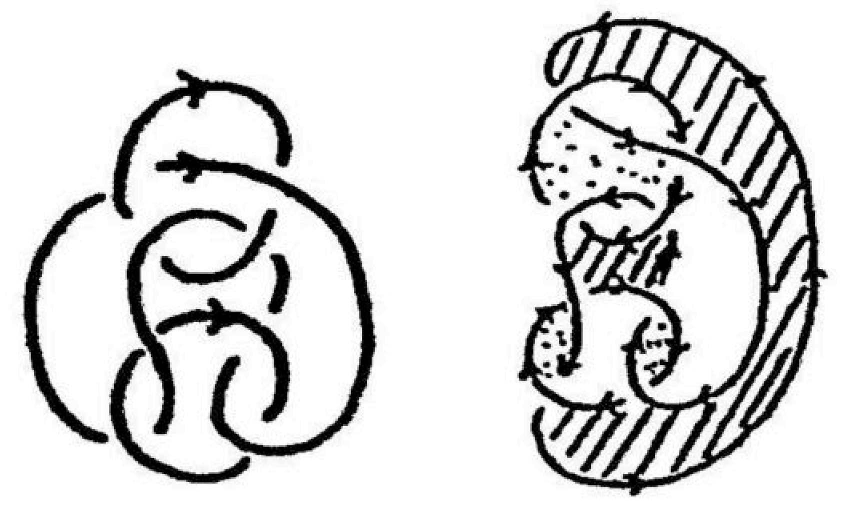
$8^3_9$②



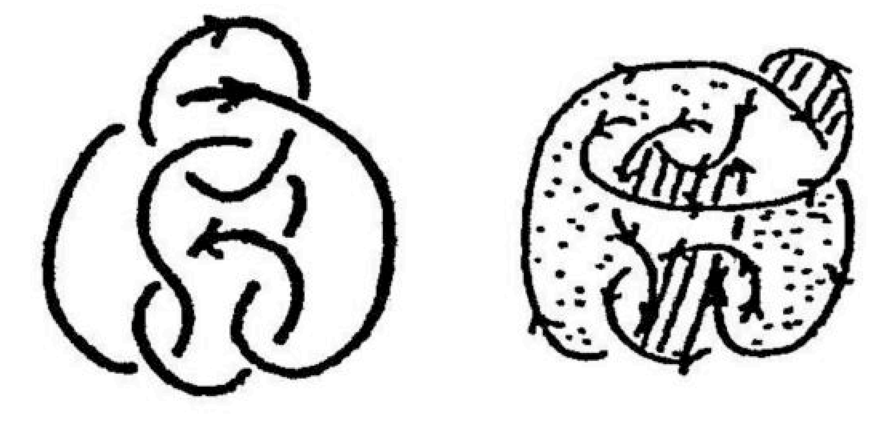
$8^3_9$③



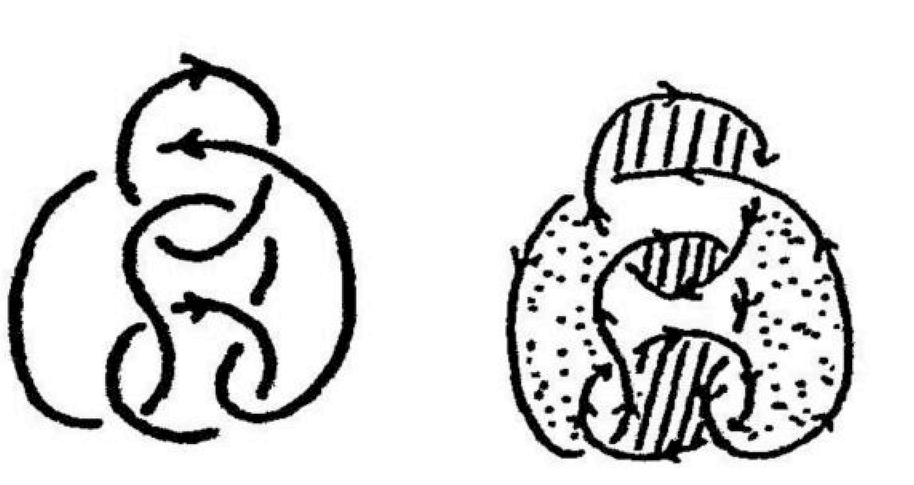
$8^3_9$④



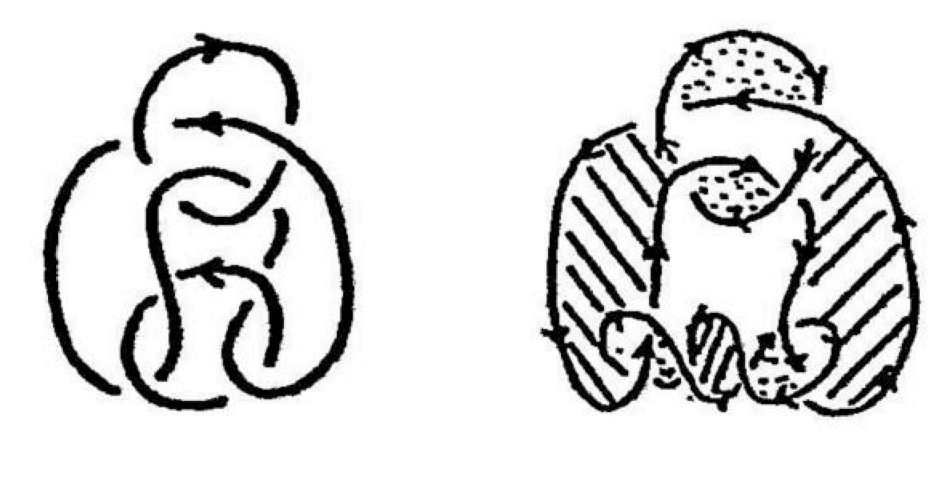
8^3_{10}①



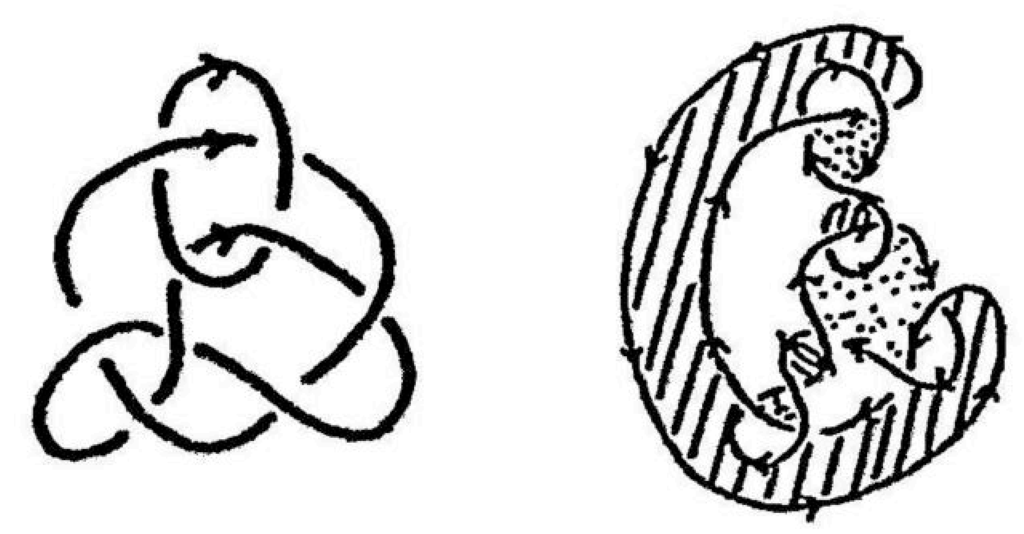
8^3_{10}②



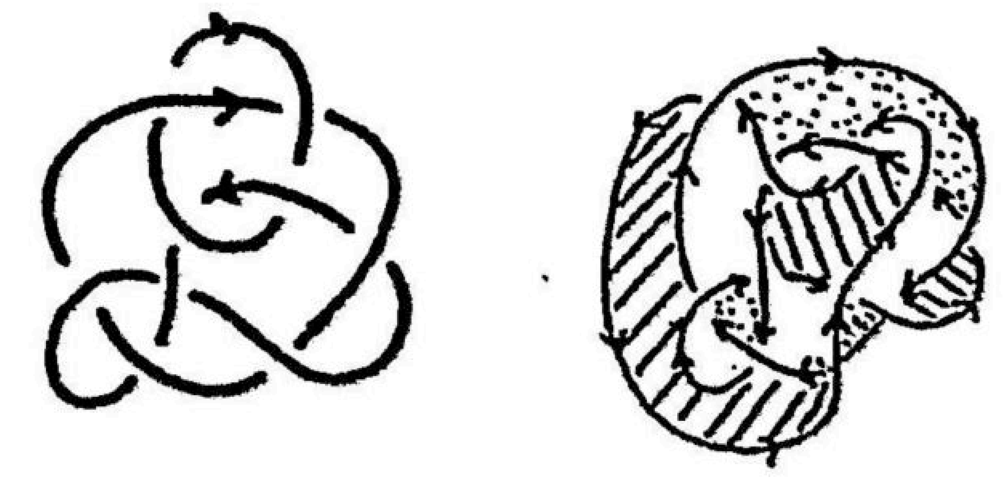
8^3_{10}③



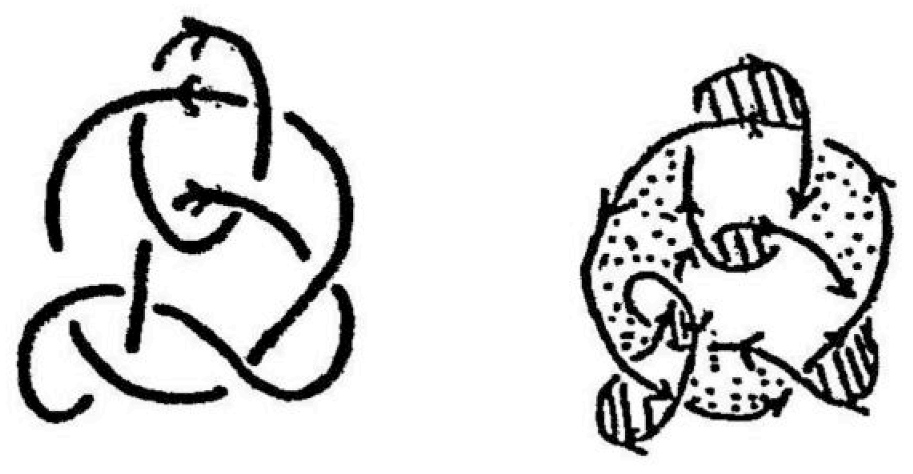
8^3_{10}④



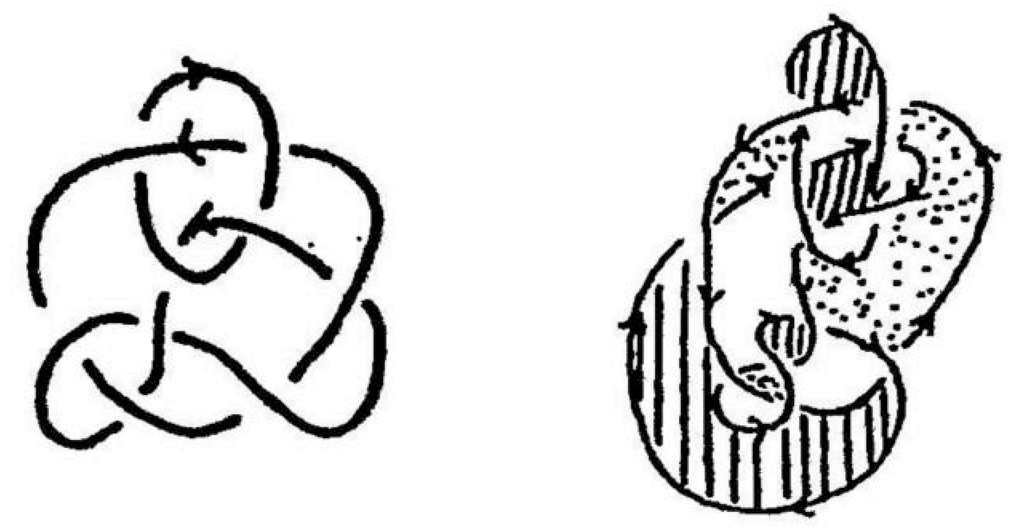
$9^3_1$①



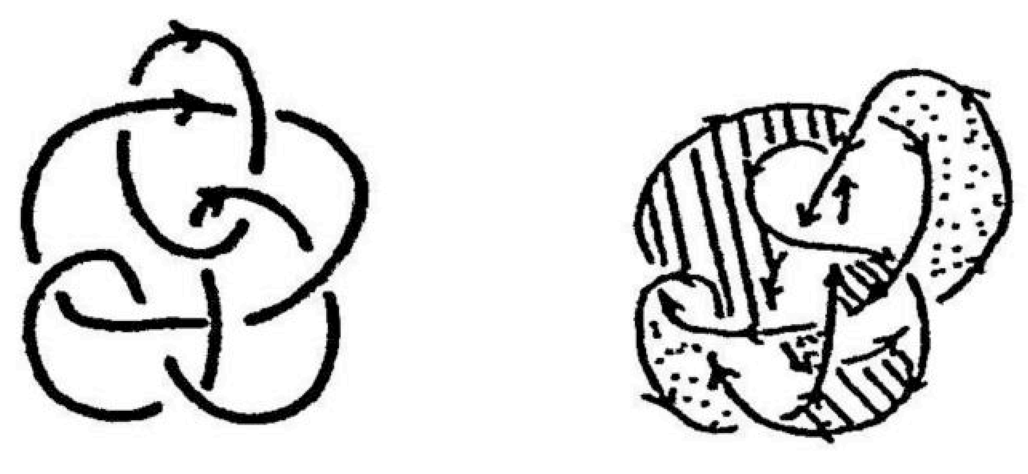
$9^3_1$②



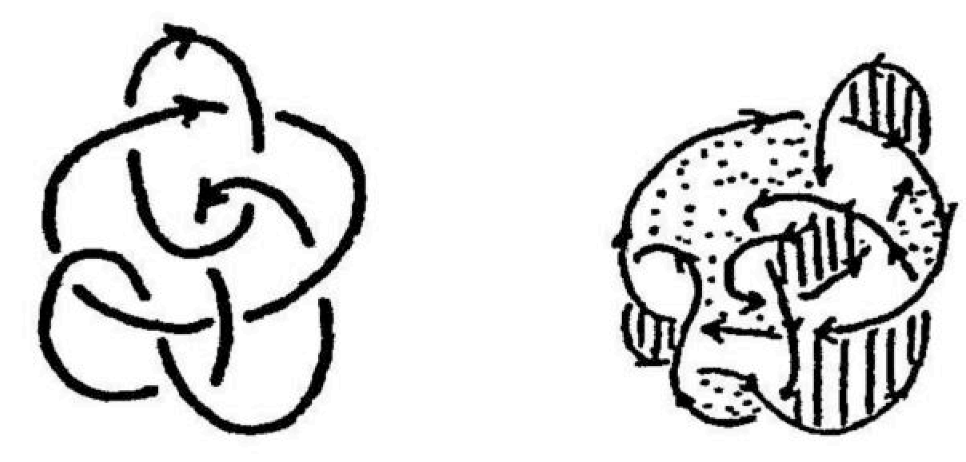
$9^3_1$③



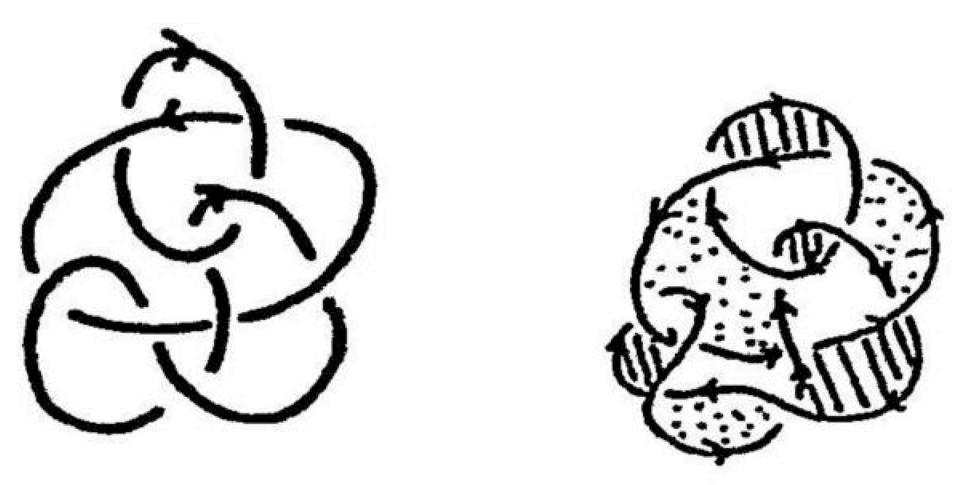
$9^3_1$④



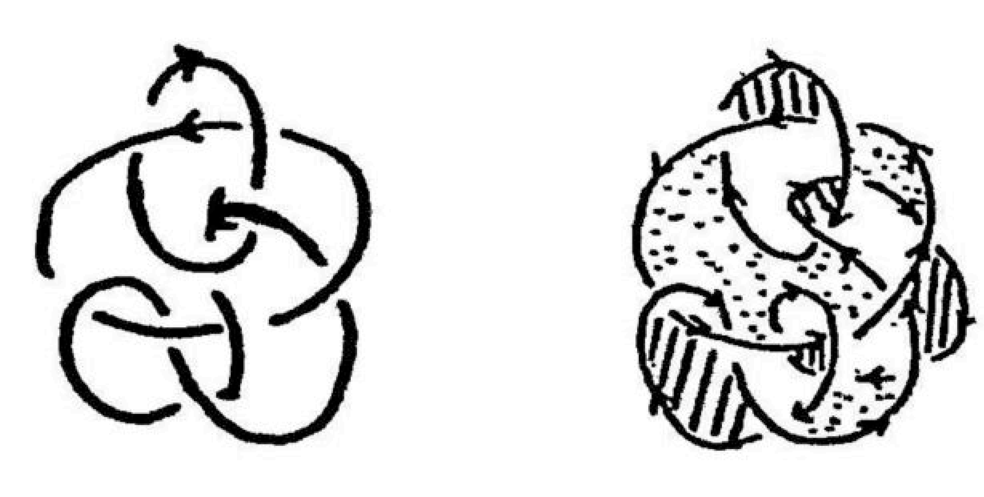
$9^3_2$①



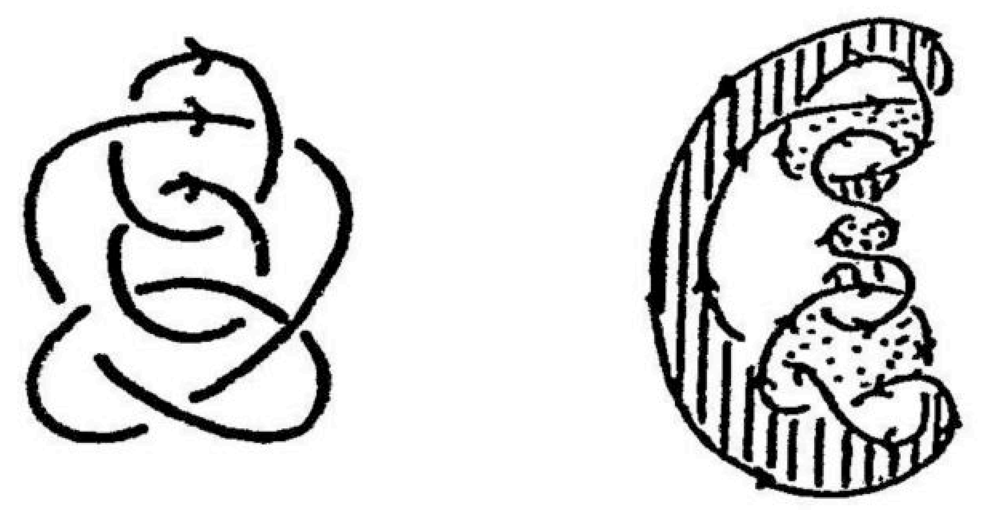
$9^3_2$②



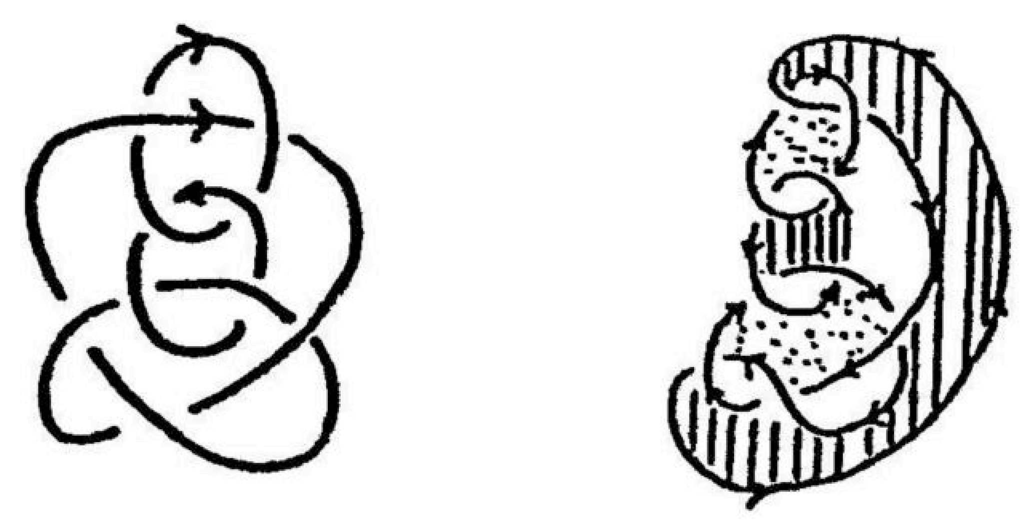
$9^3_2$③



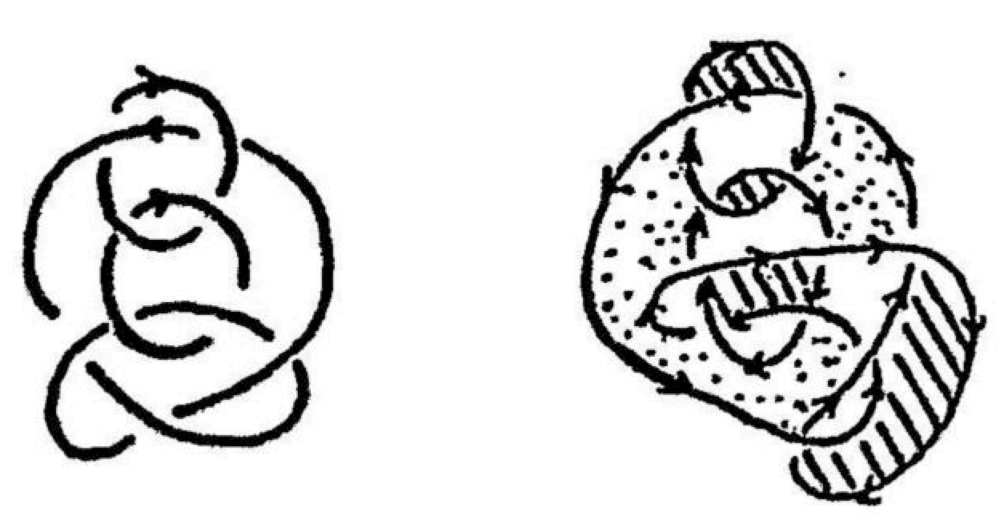
$9^3_2$④



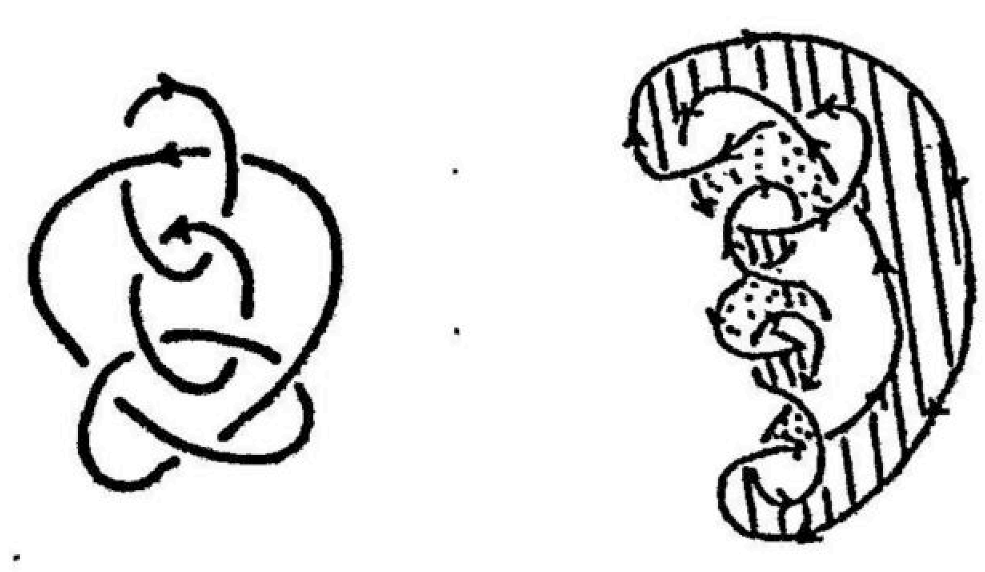
$9^3_3$①



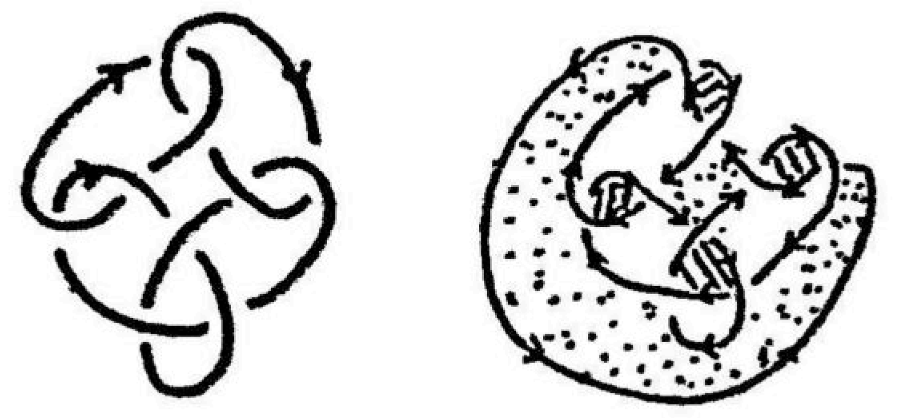
$9^3_3$②



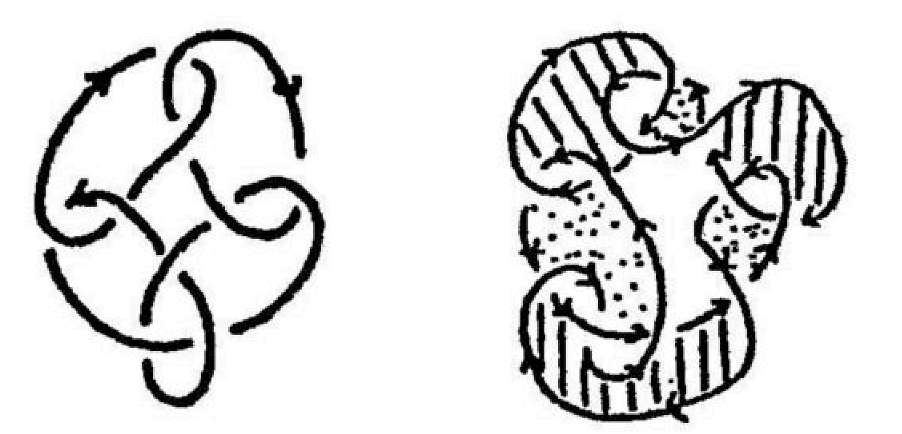
$9^3_3$③



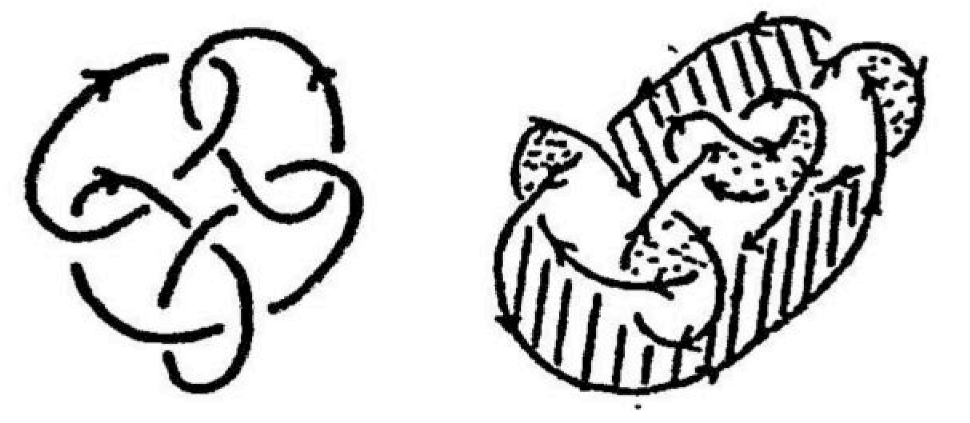
$9^3_3$④



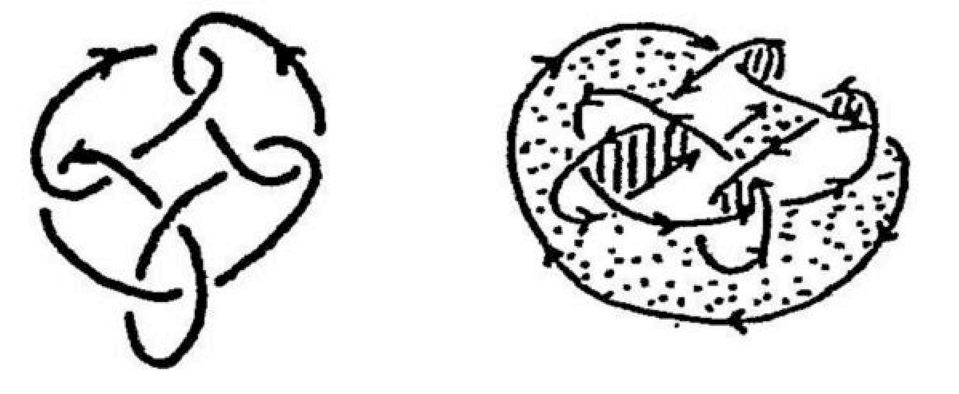
$9^3_4$①



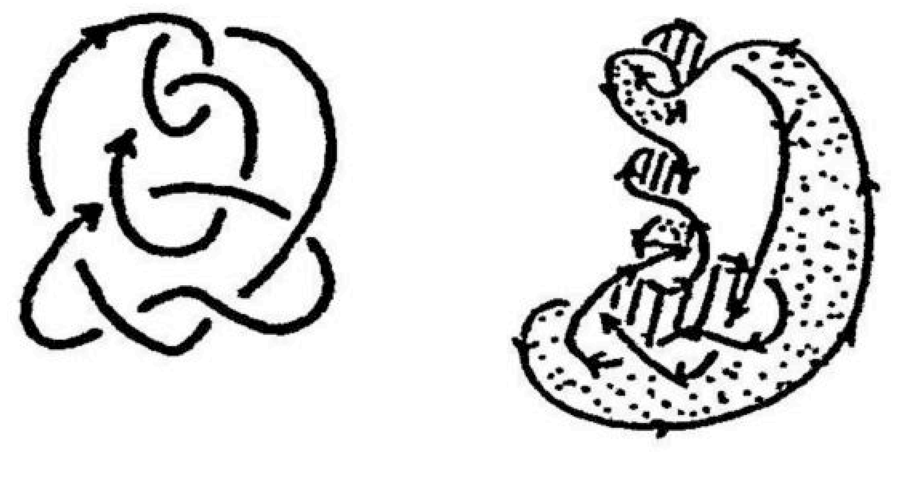
$9^3_4$②



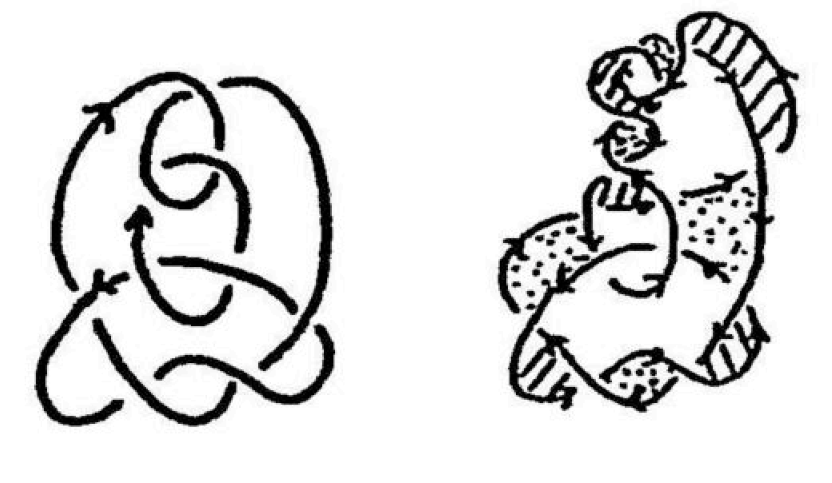
$9^3_4$③



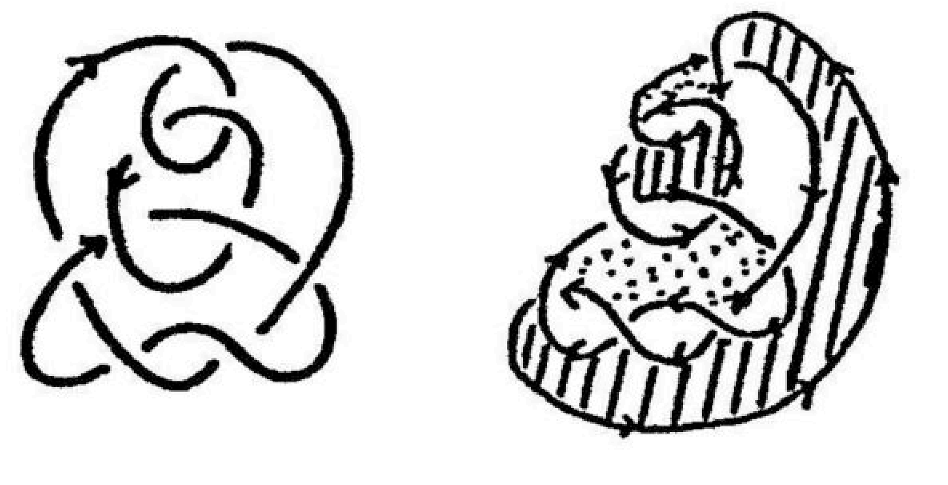
$9^3_4$④



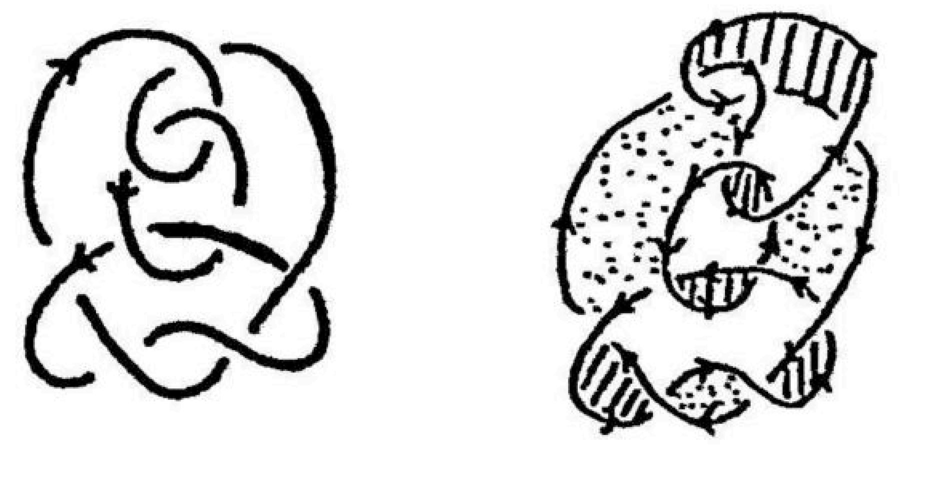
$9^3_5$①



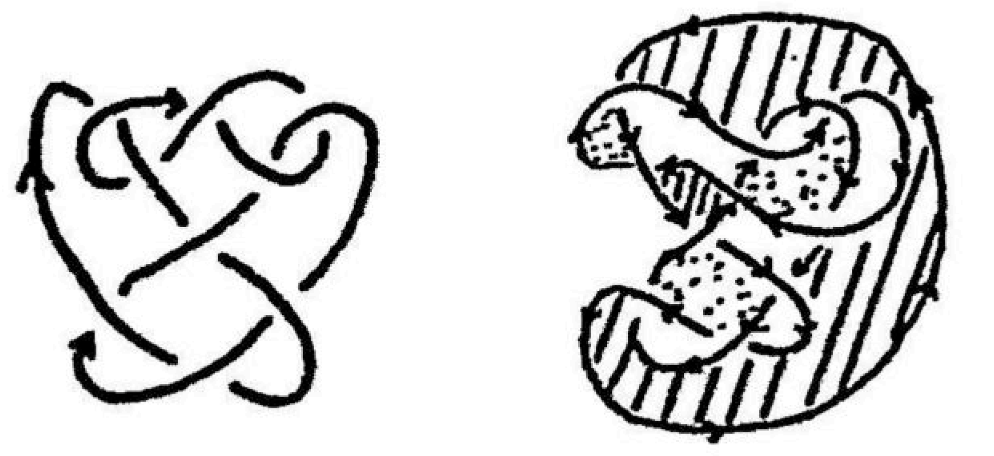
$9^3_5$②



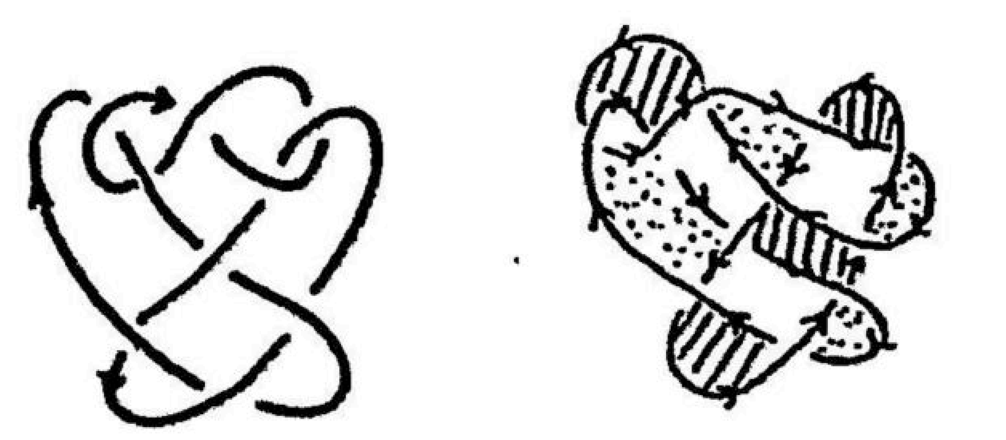
$9^3_5$③



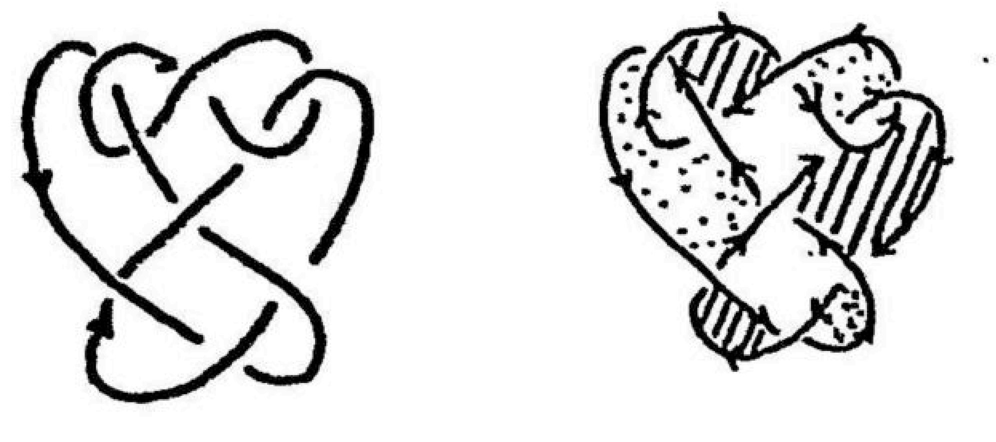
$9^3_5$④



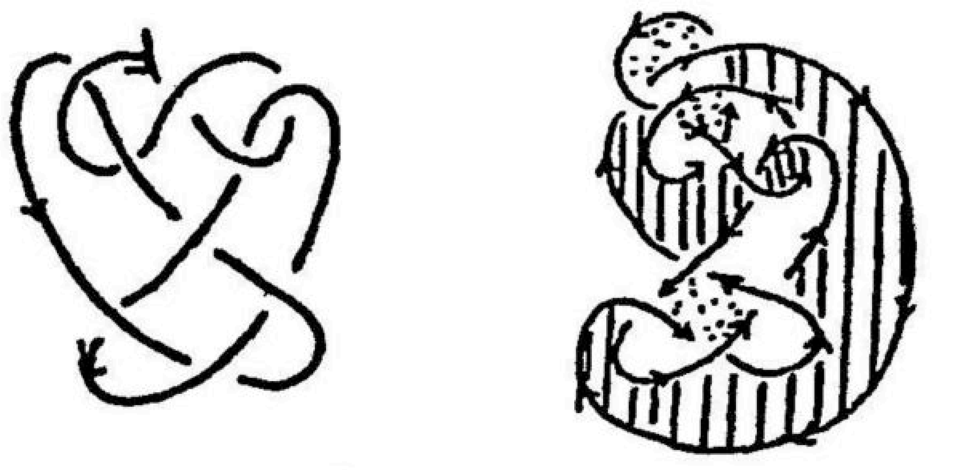
$9^3_6$①



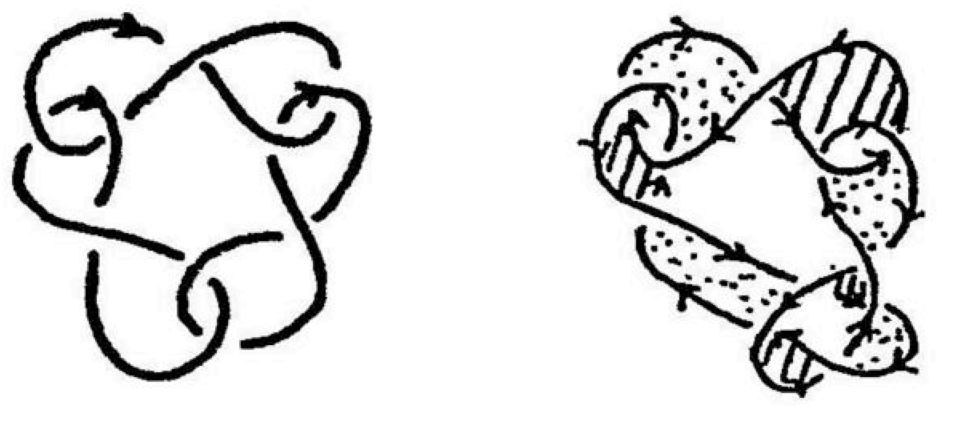
$9^3_6$②



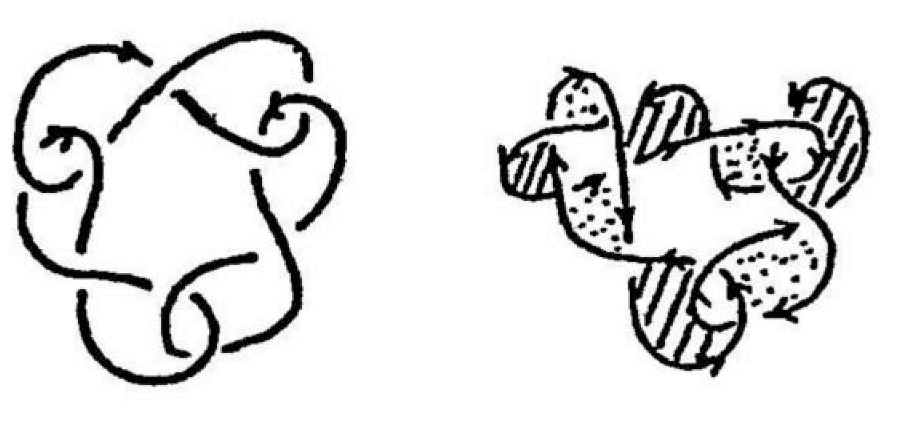
$9^3_6$③



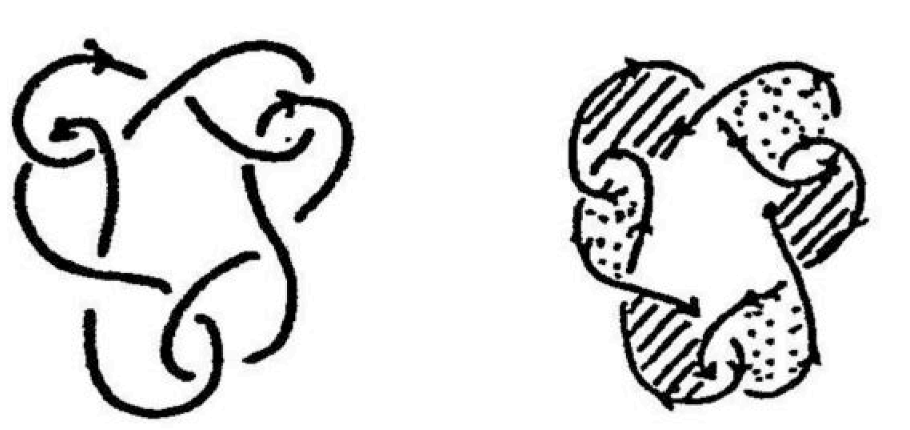
$9^3_6$④



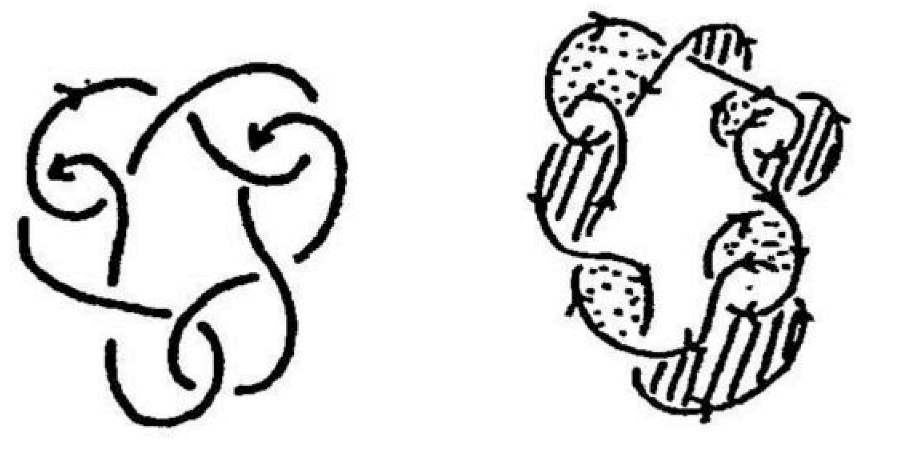
$9^3_7$①



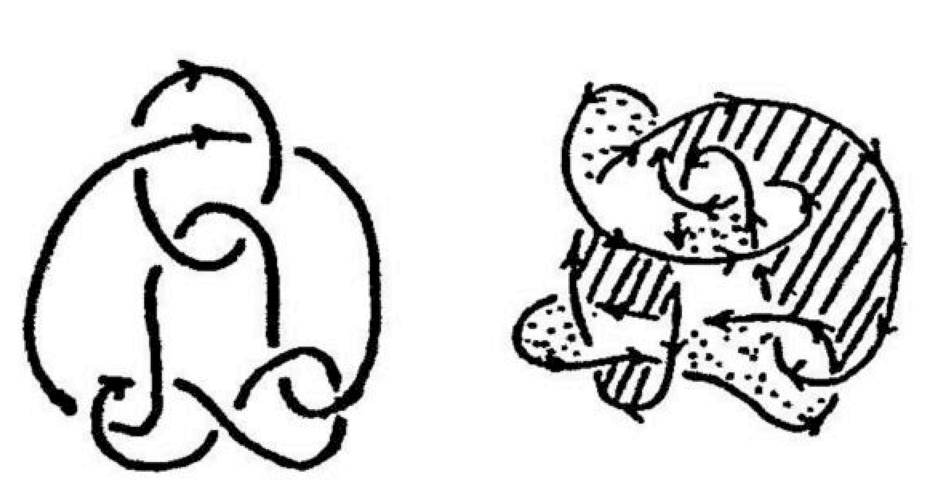
$9^3_7$②



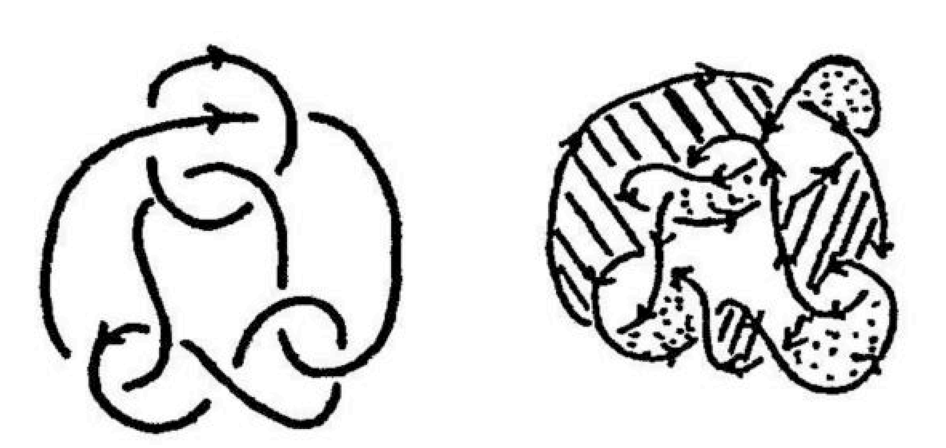
$9^3_7$③



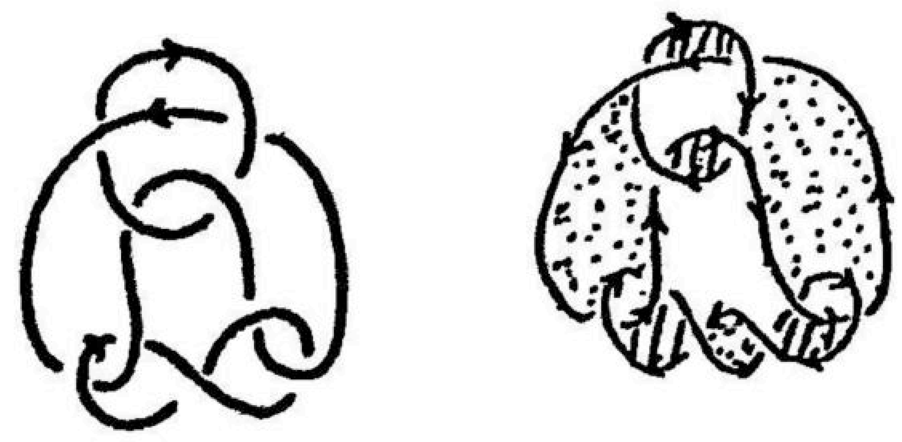
$9^3_7$④



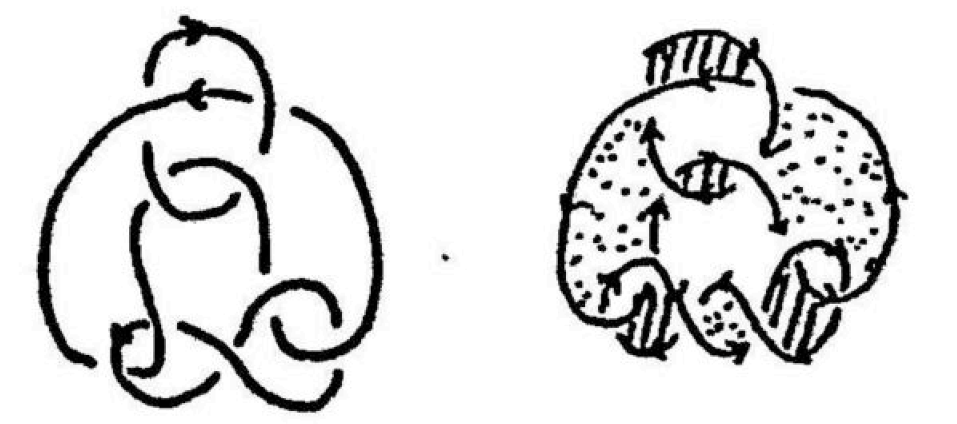
$9^3_8$①



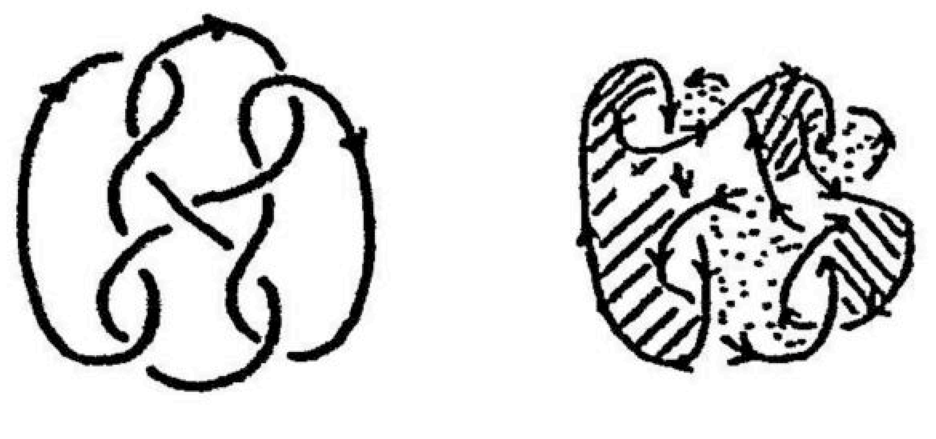
$9^3_8$②



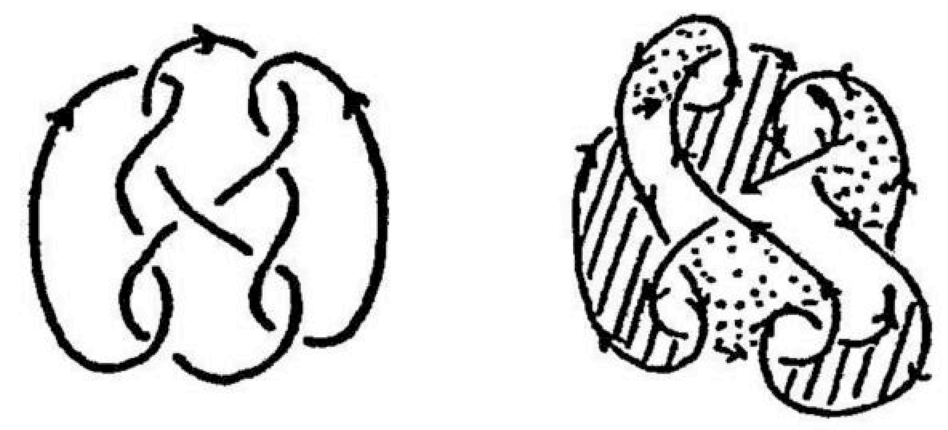
$9^3_8$③



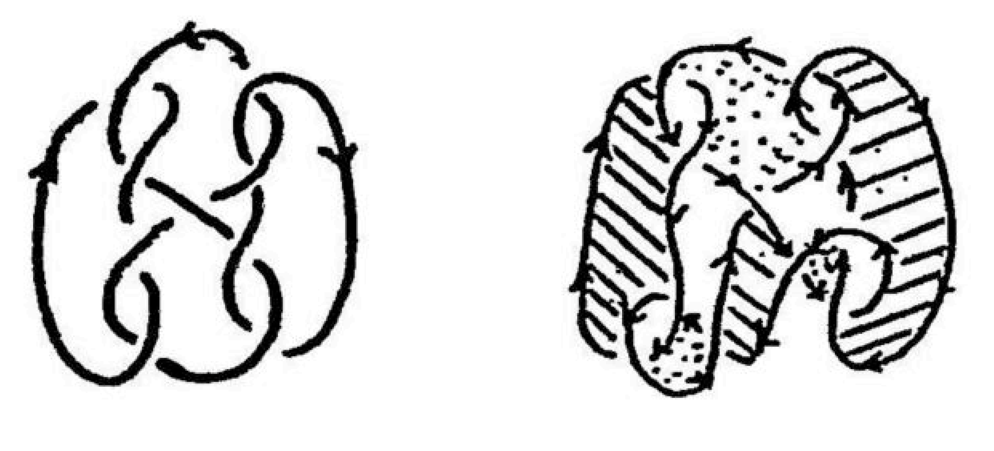
$9^3_8$④



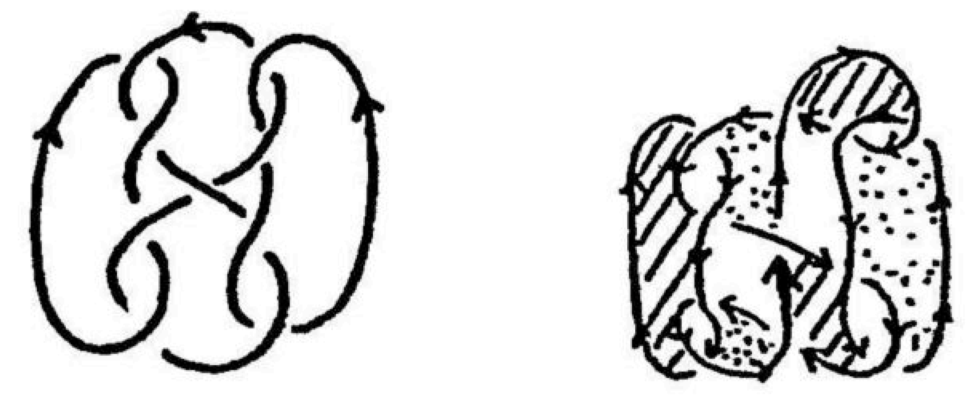
$9^3_9$①



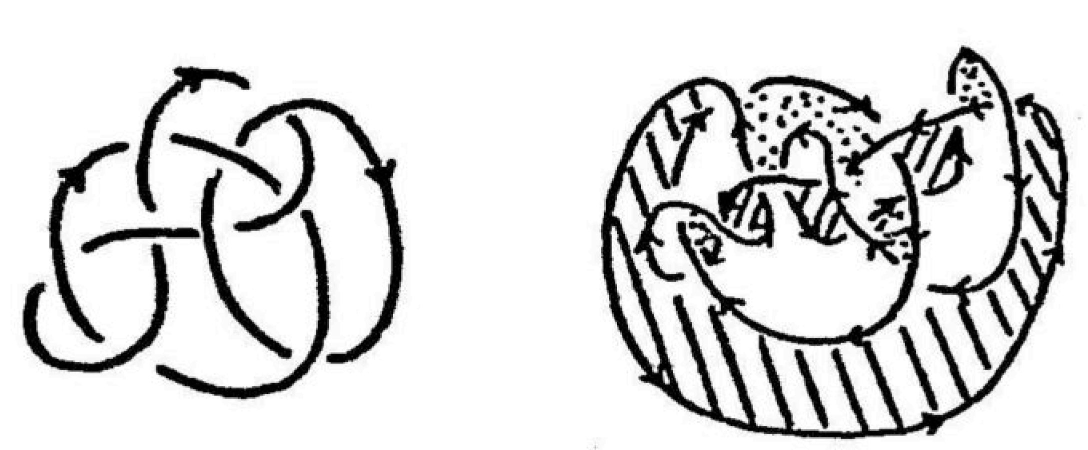
$9^3_9$②



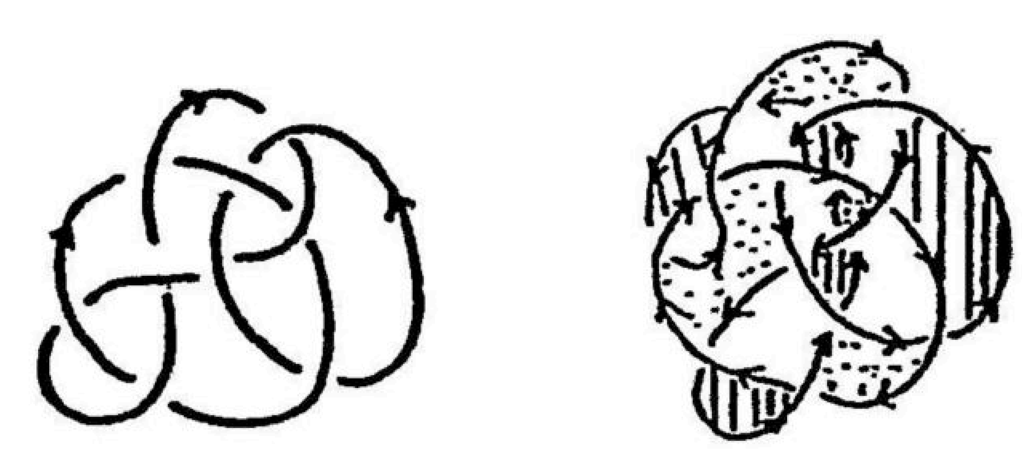
$9^3_9$③



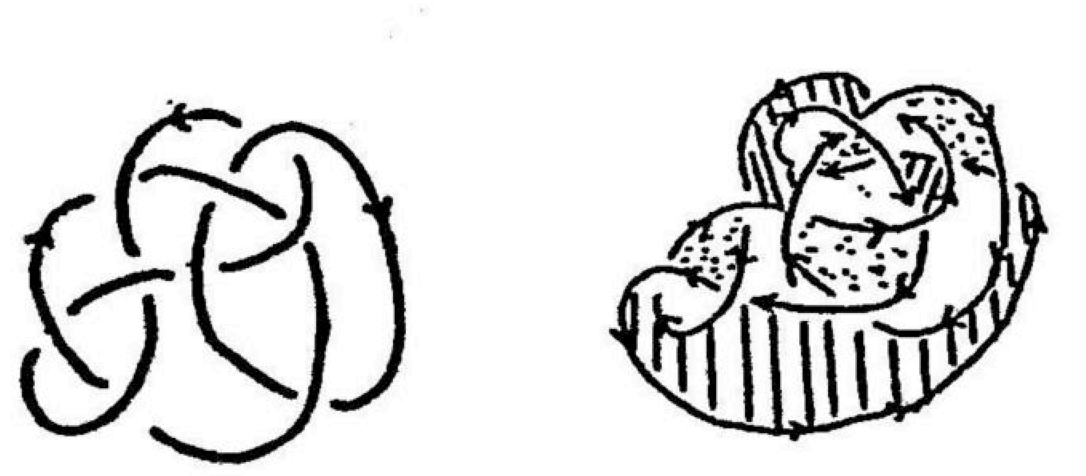
$9^3_9$④



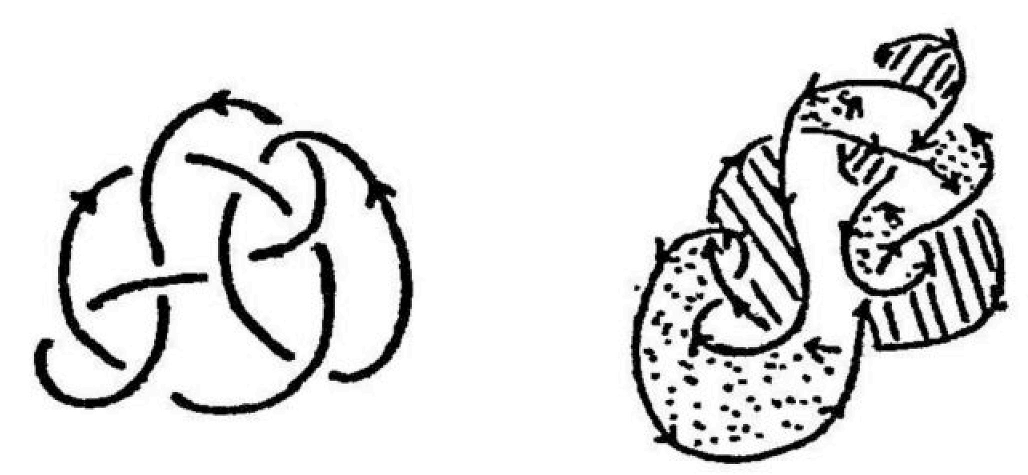
9^3_{10}①



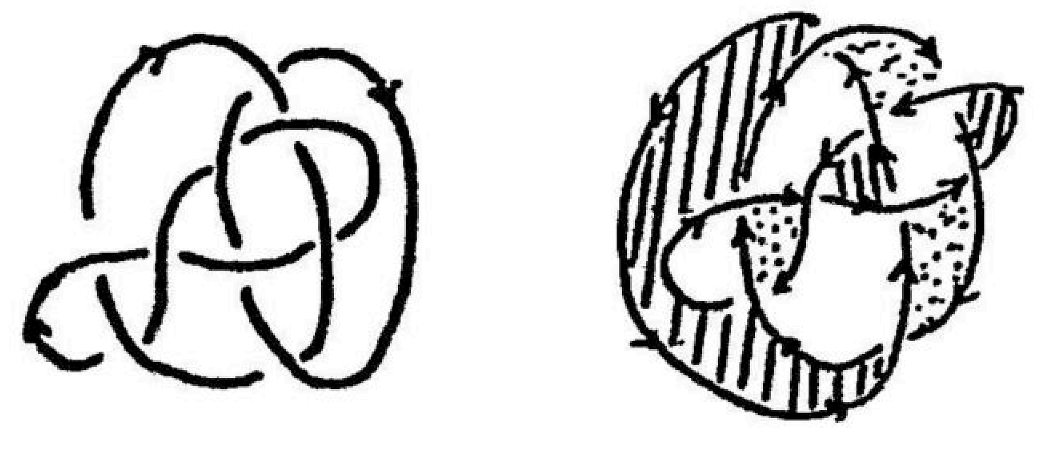
9^3_{10}②



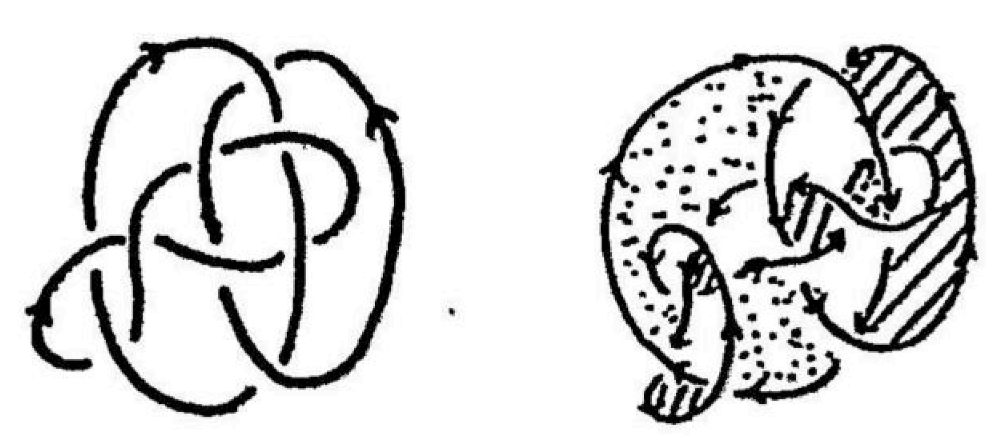
9^3_{10}③



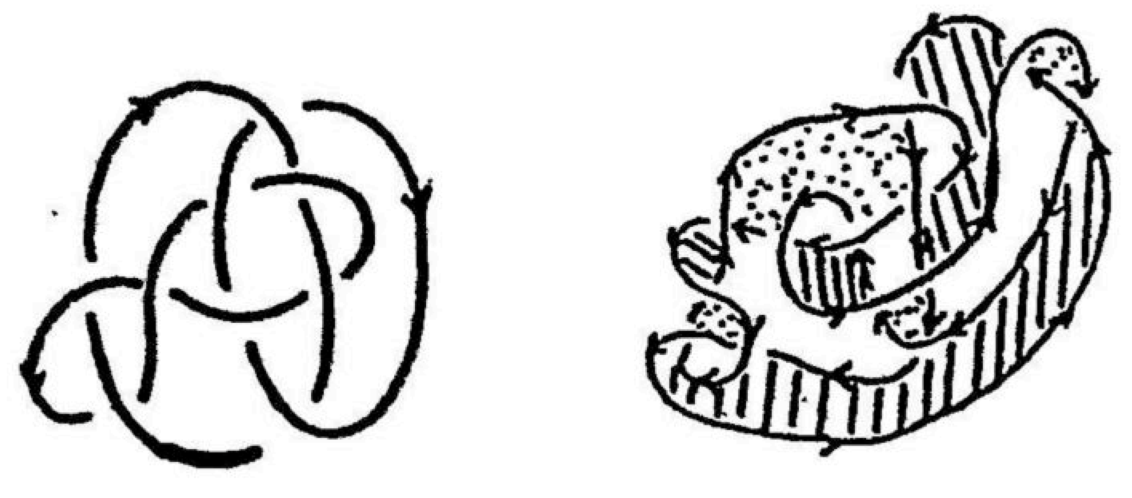
9^3_{10}④



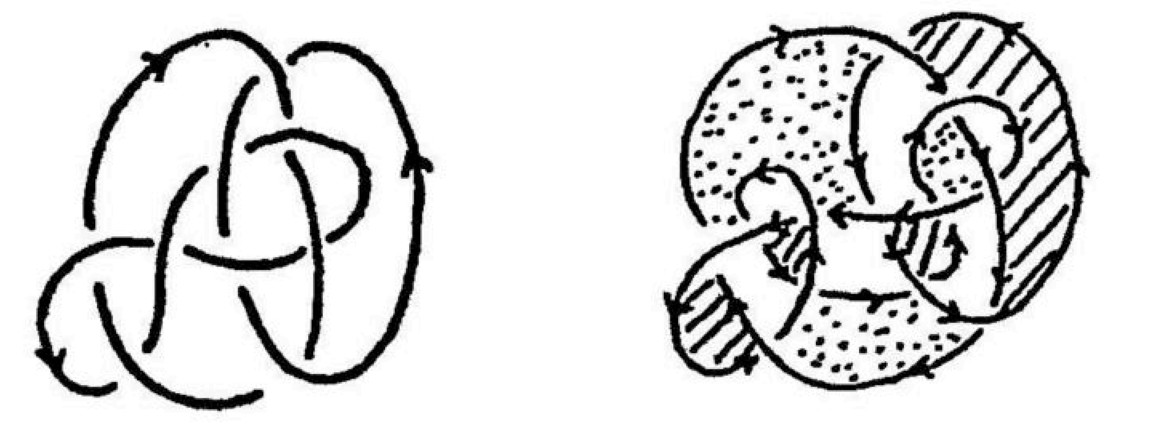
9^3_{11}①



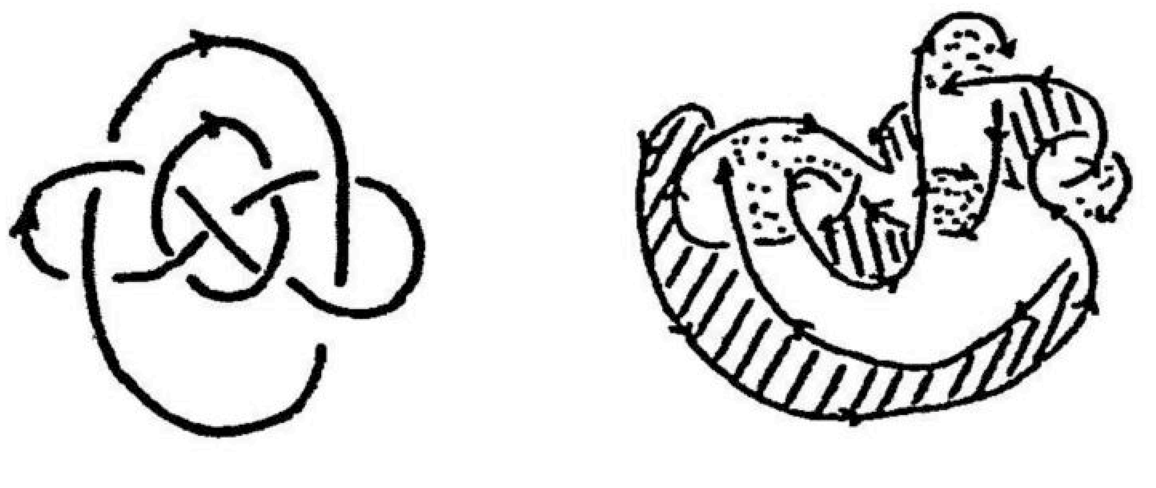
9^3_{11}②



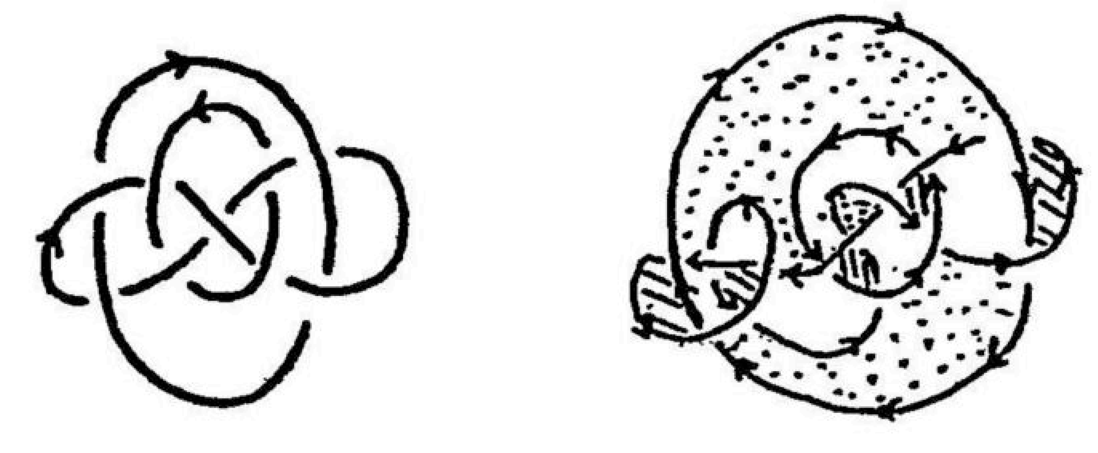
9^3_{11}③



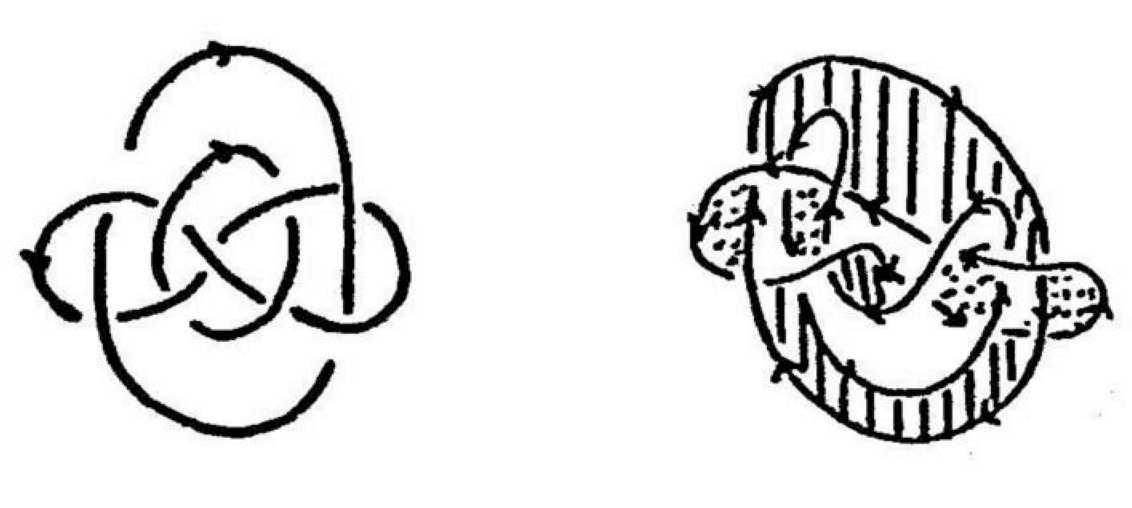
9^3_{11}④



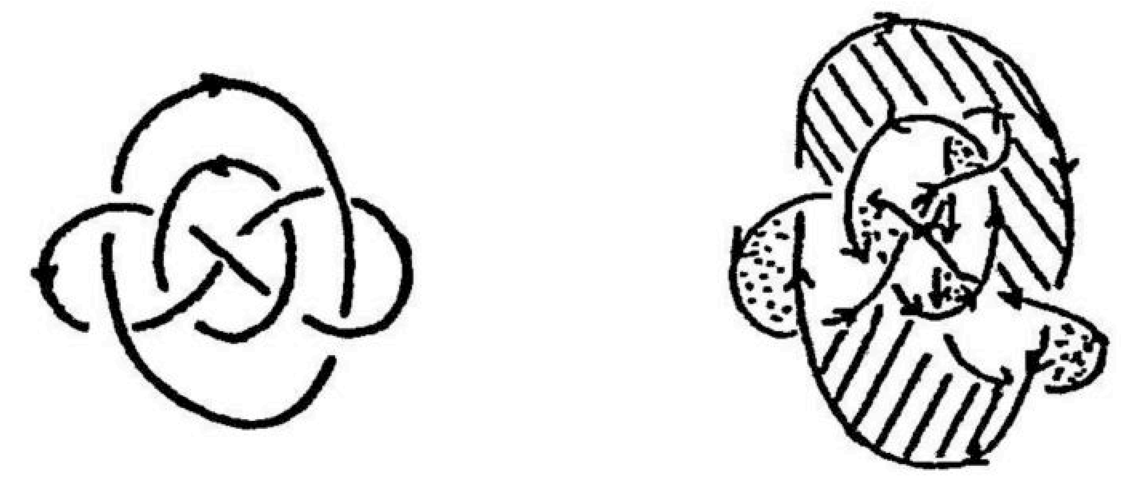
9^3_{12}①



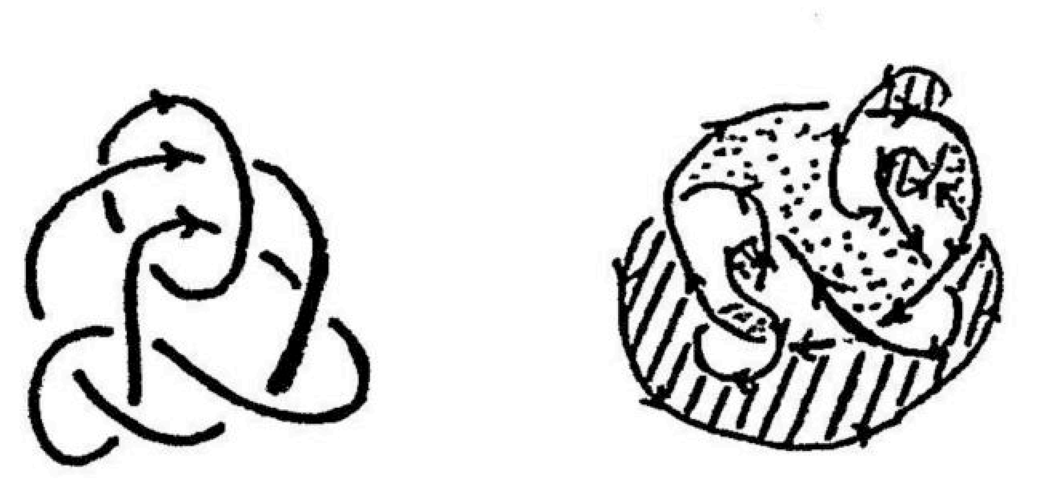
9^3_{12}②



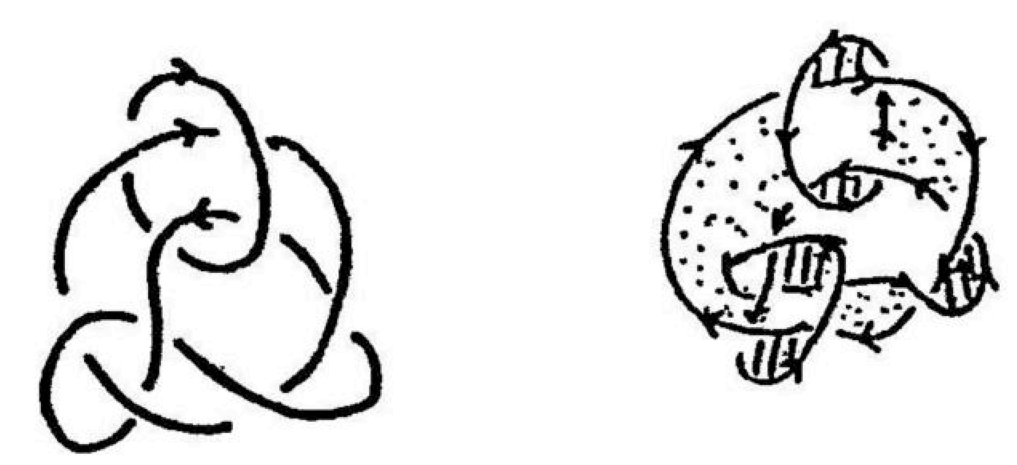
9^3_{12}③



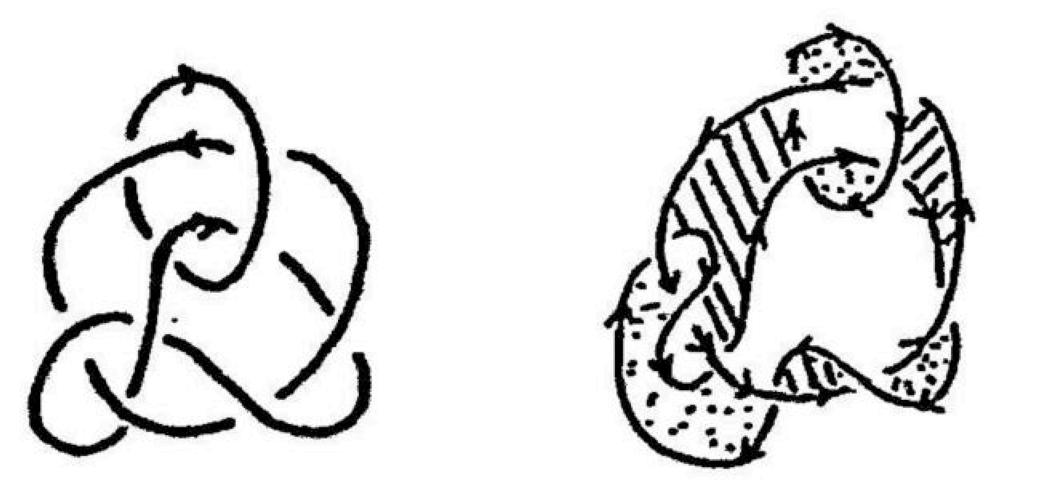
9^3_{12}④



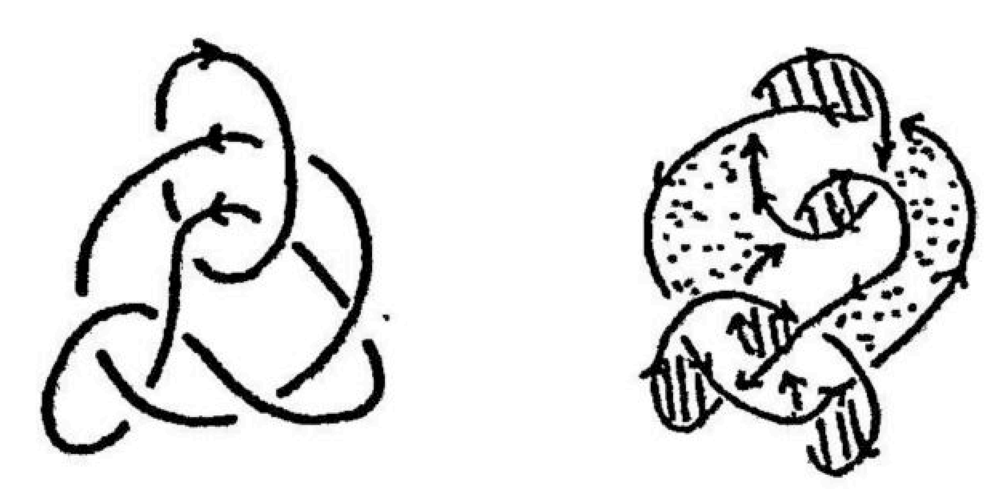
9^3_{13}①



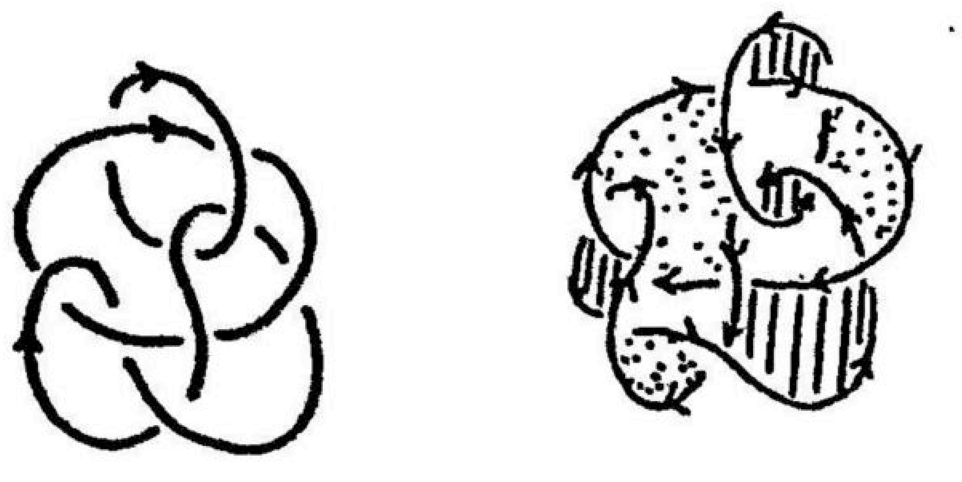
9^3_{13}②



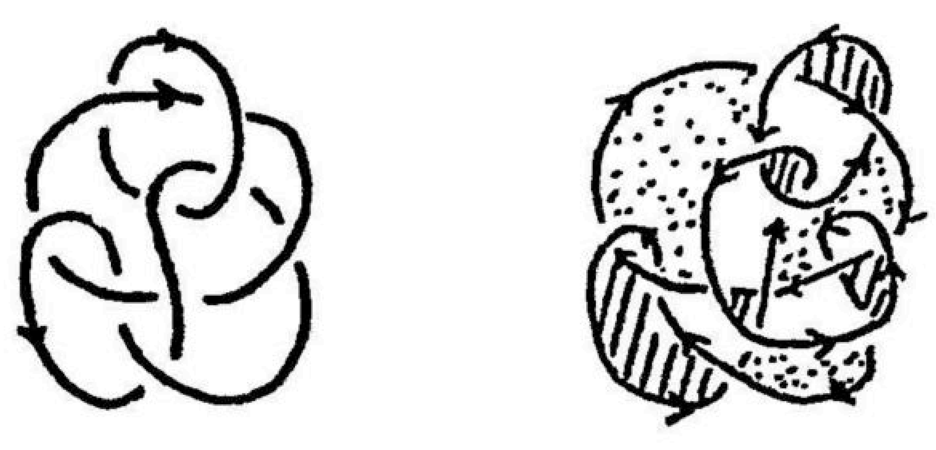
9^3_{13}③



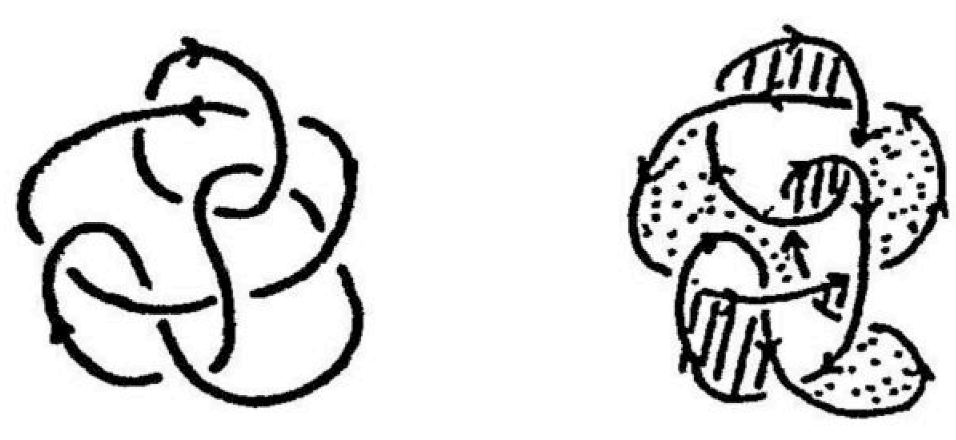
9^3_{13}④



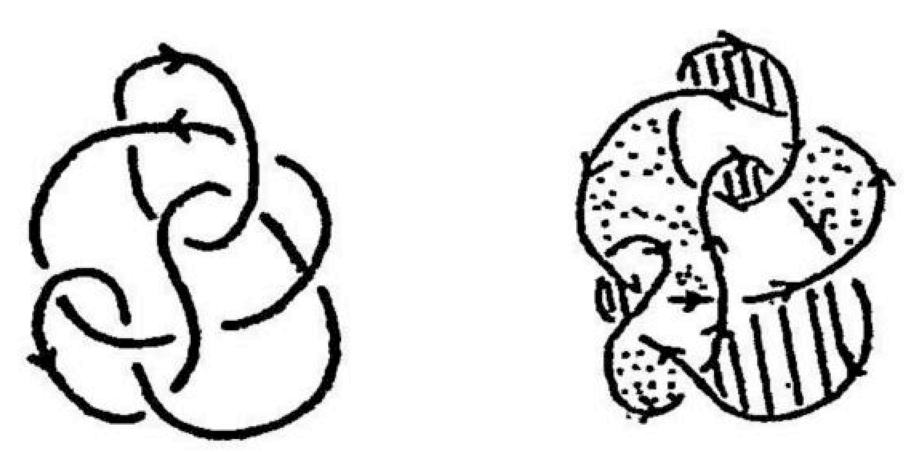
9^3_{14}①



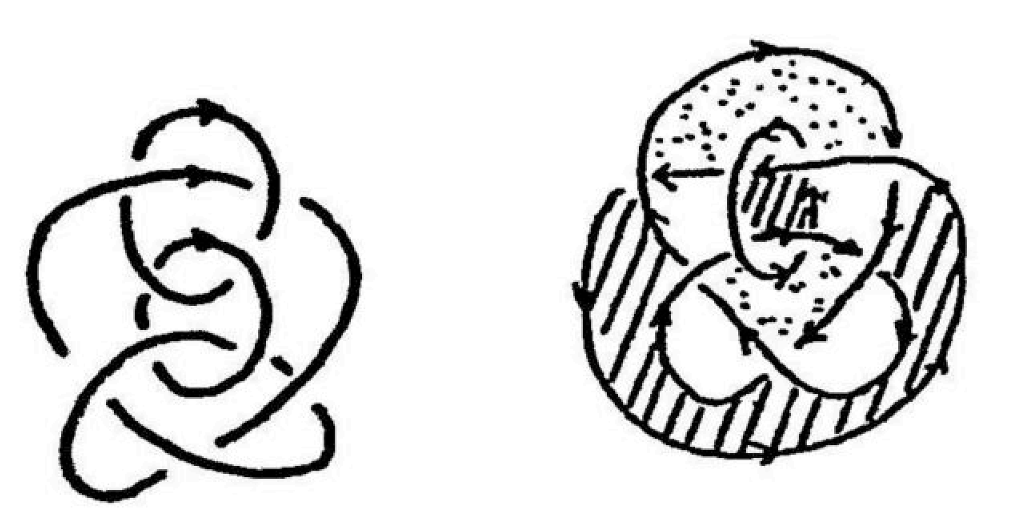
9^3_{14}②



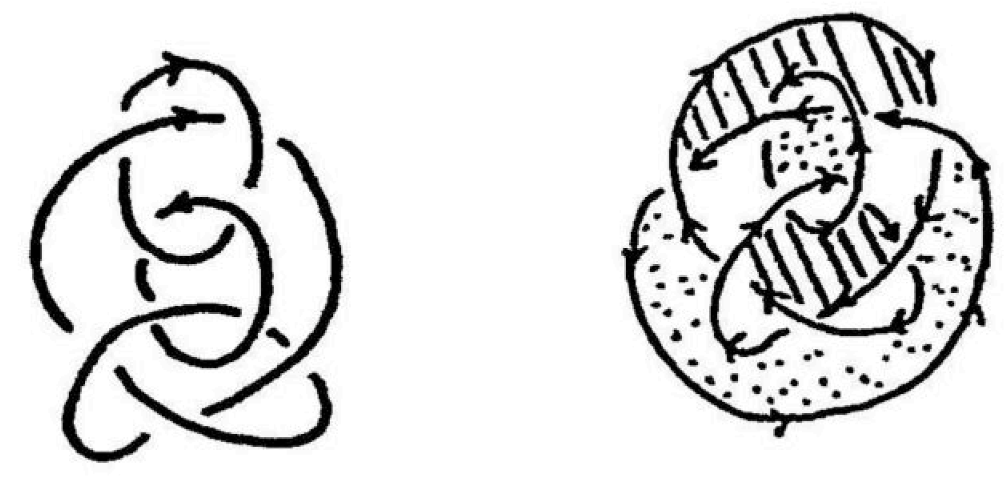
9^3_{14}③



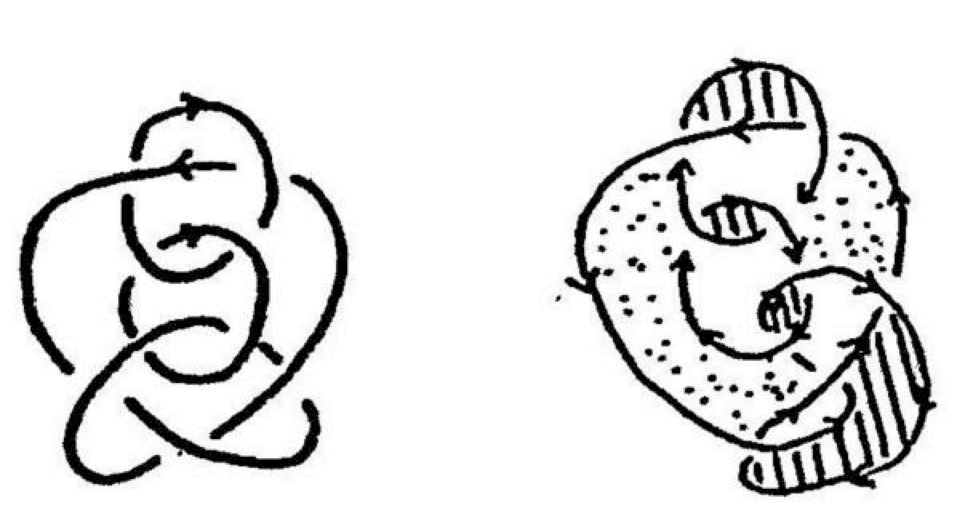
9^3_{14}④



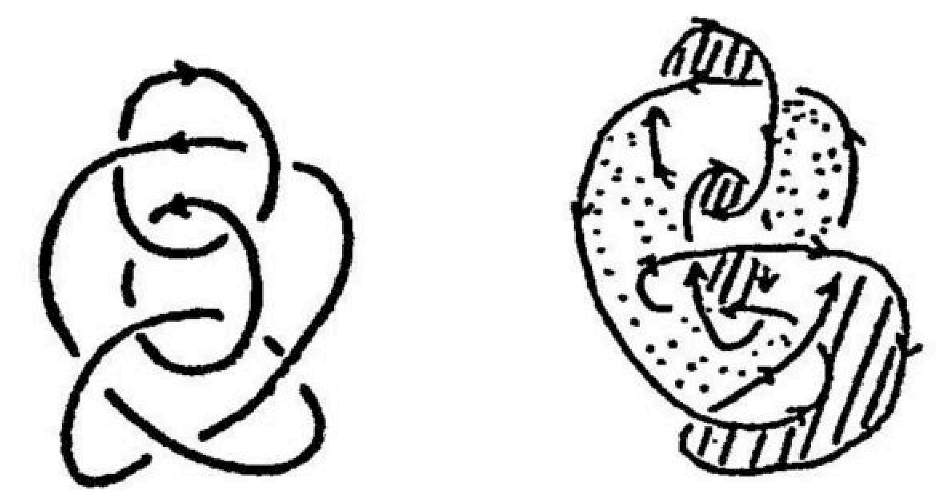
9^3_{15}①



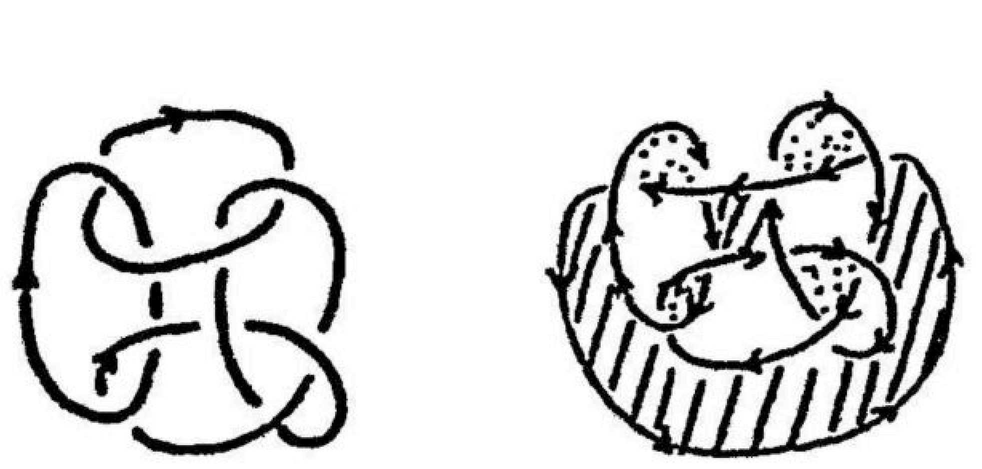
9^3_{15}②



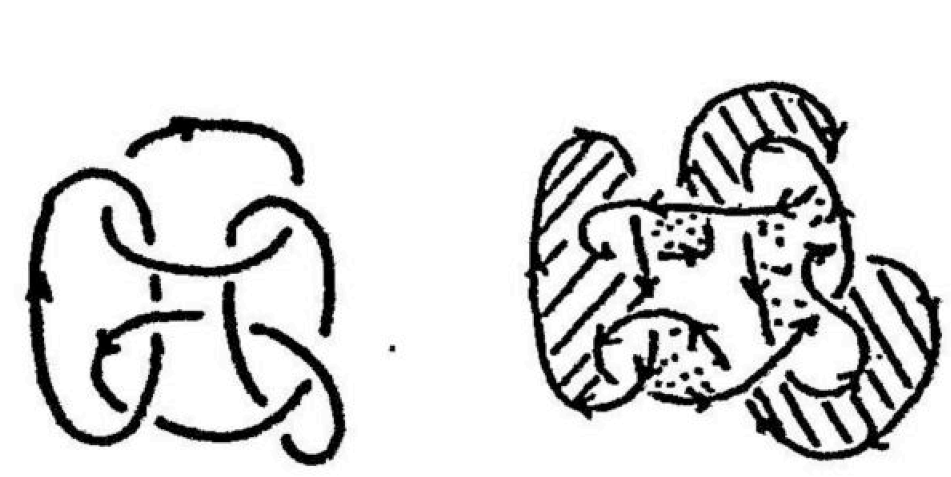
9^3_{15}③



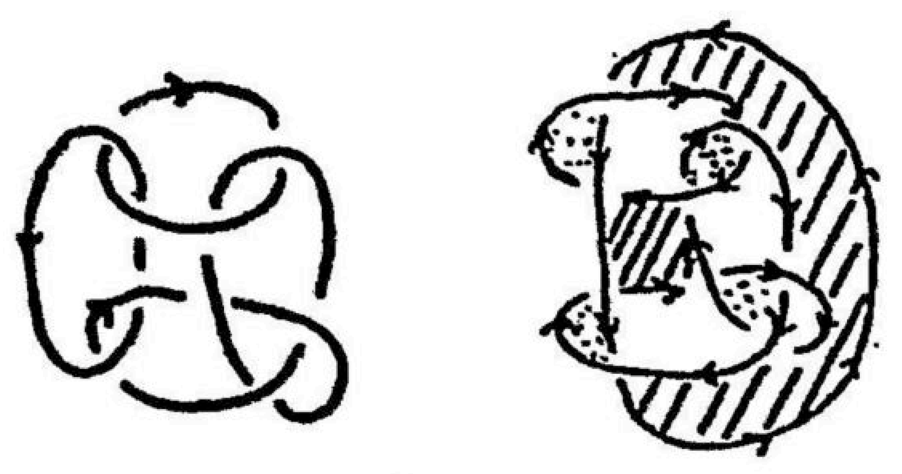
9^3_{15}④



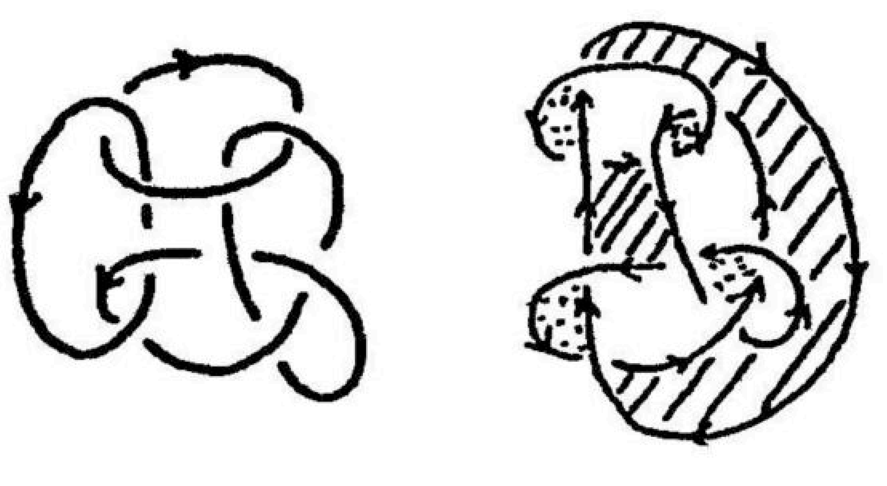
9^3_{16}①



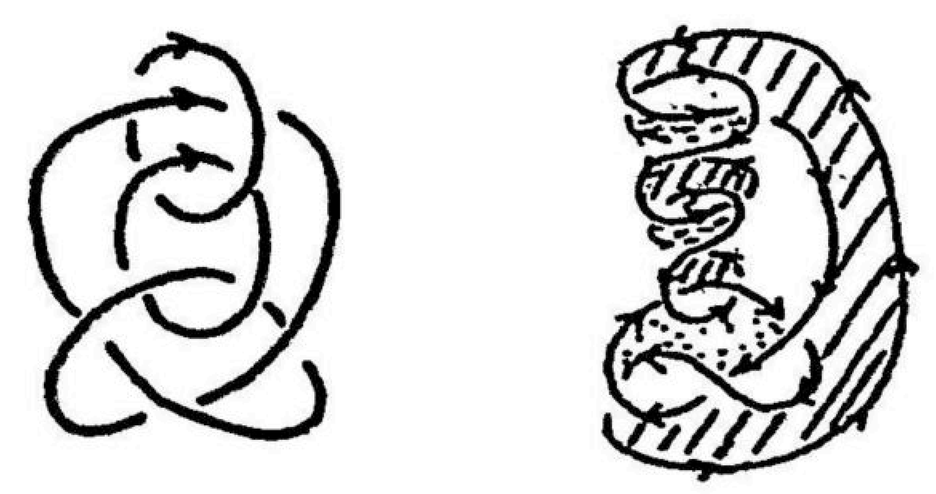
9^3_{16}②



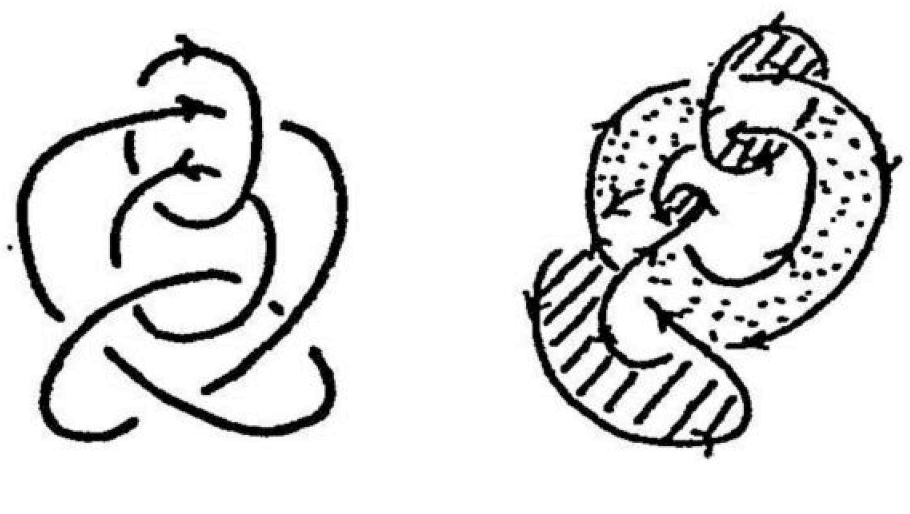
9^3_{16}③



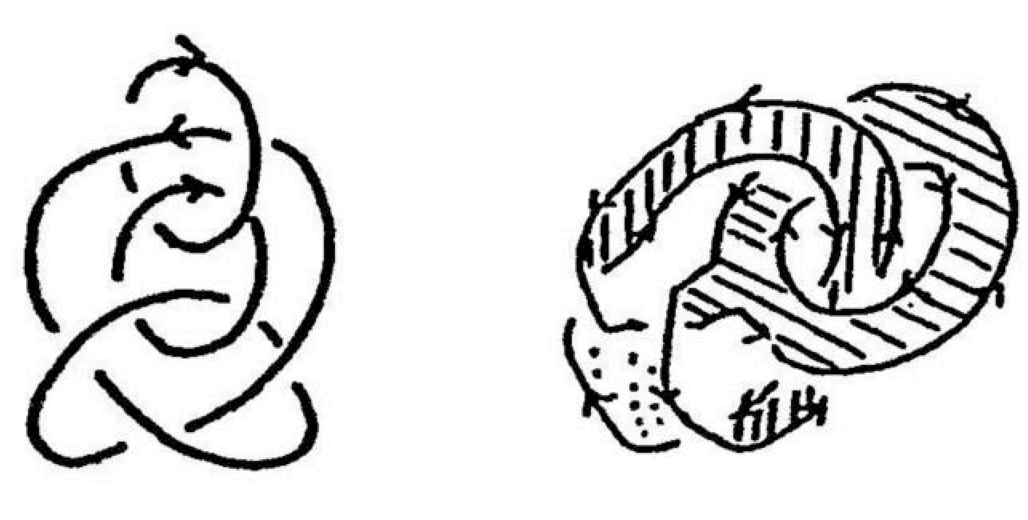
9^3_{16}④



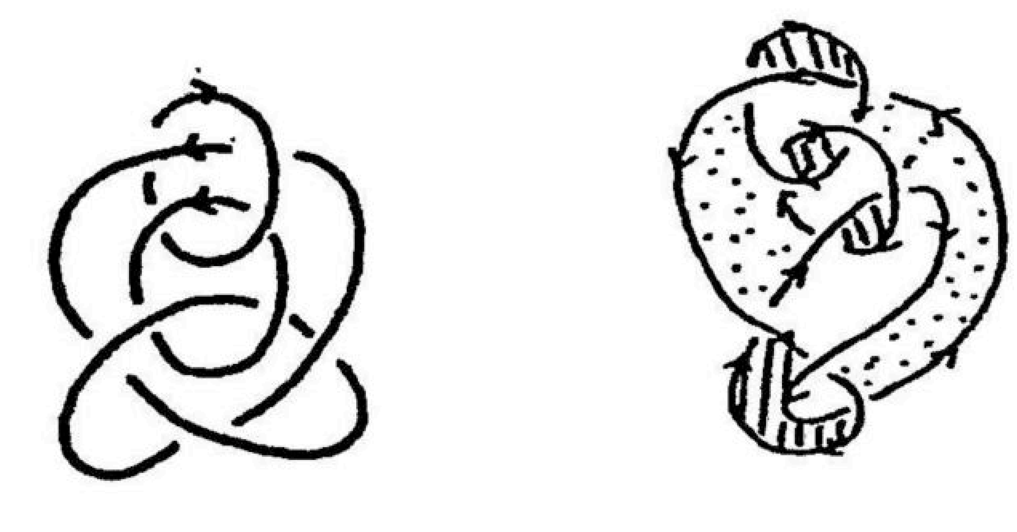
9^3_{17}①



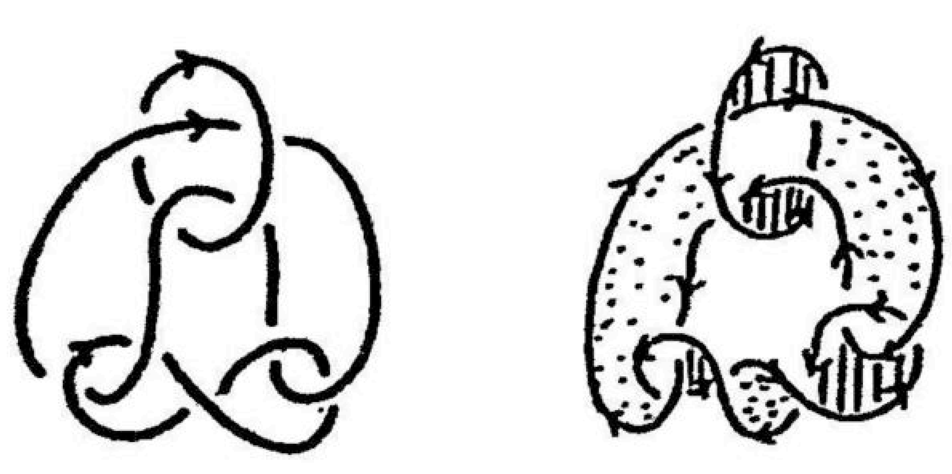
9^3_{17}②



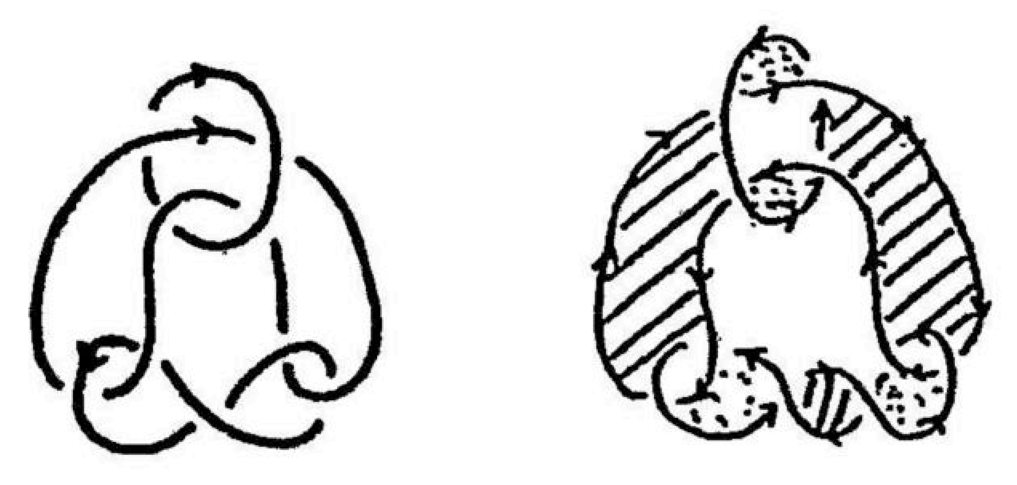
9^3_{17}③



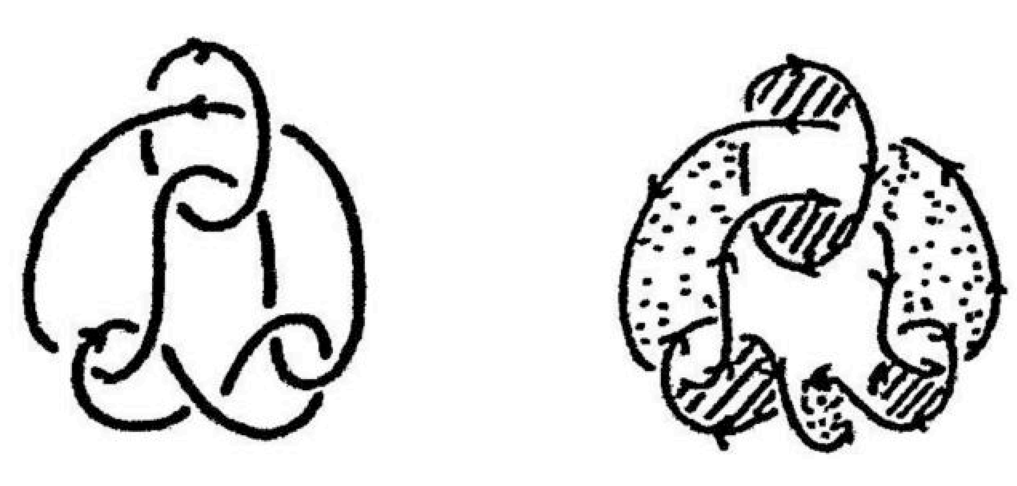
9^3_{17}④



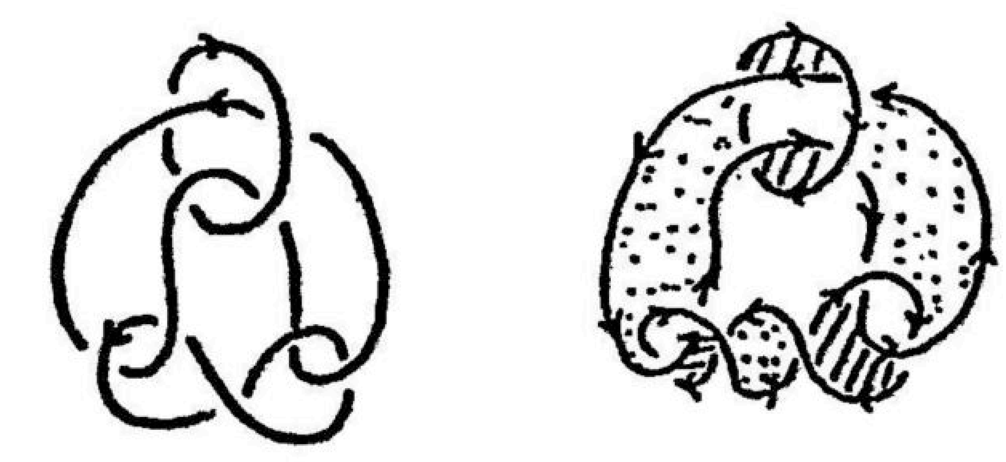
9^3_{18}①



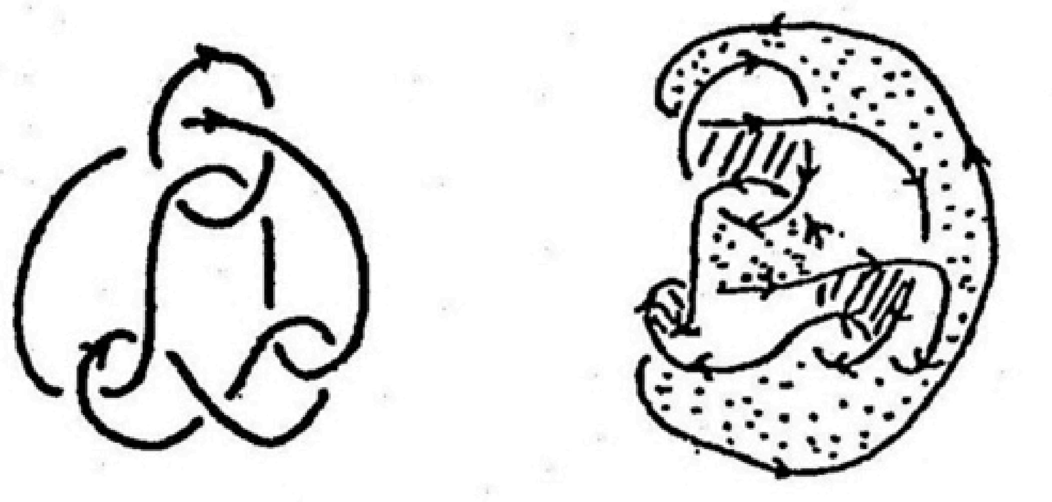
9^3_{18}②



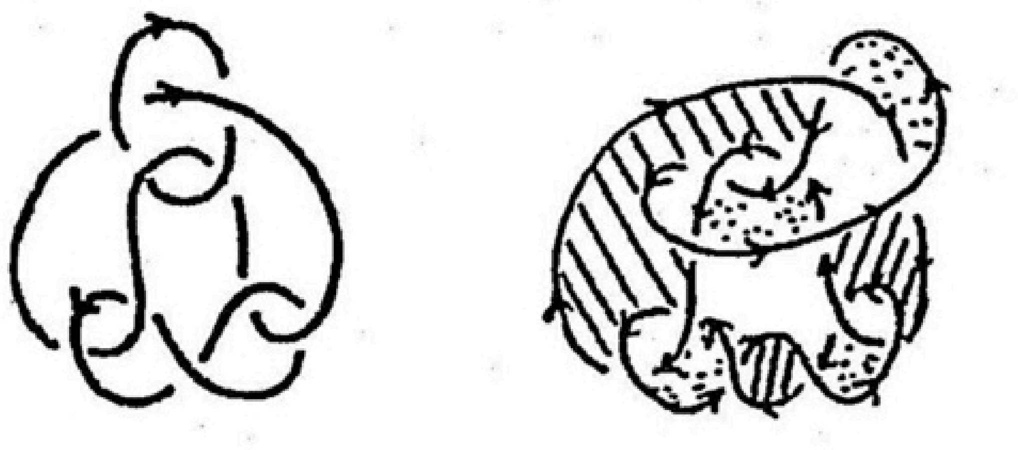
9^3_{18}③



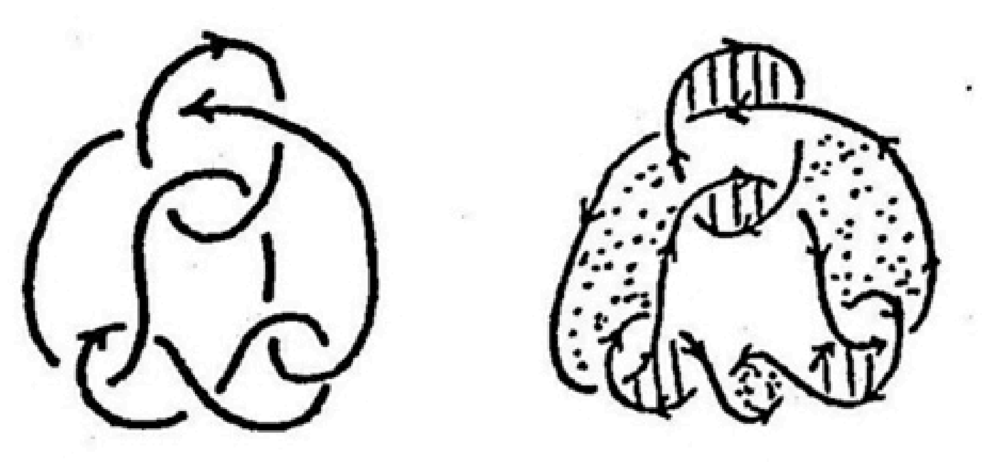
9^3_{18}④



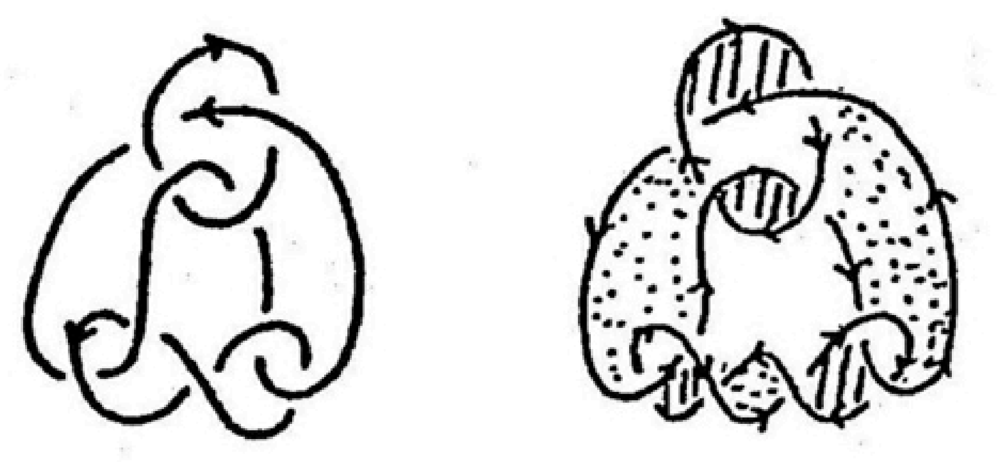
9^3_{19}①



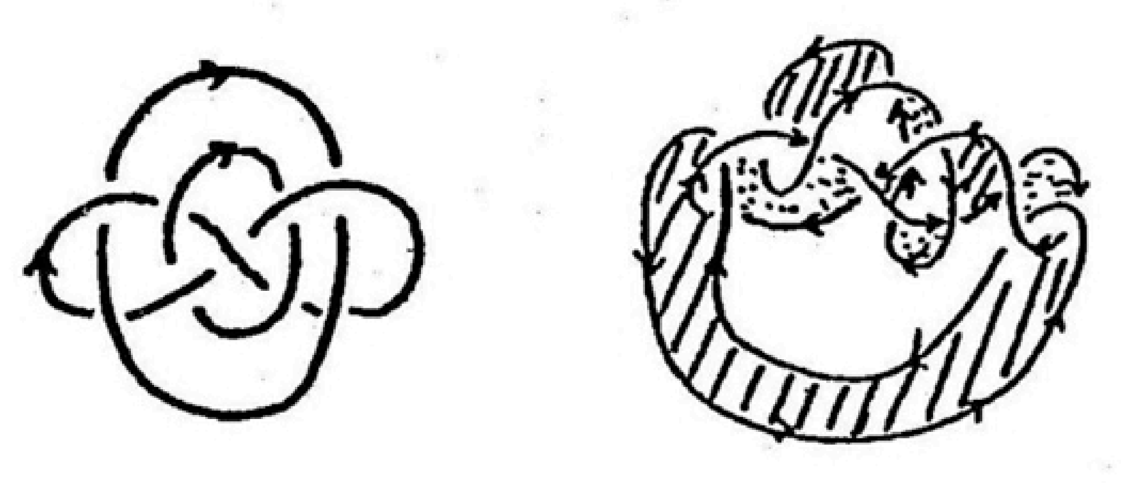
9^3_{19}②



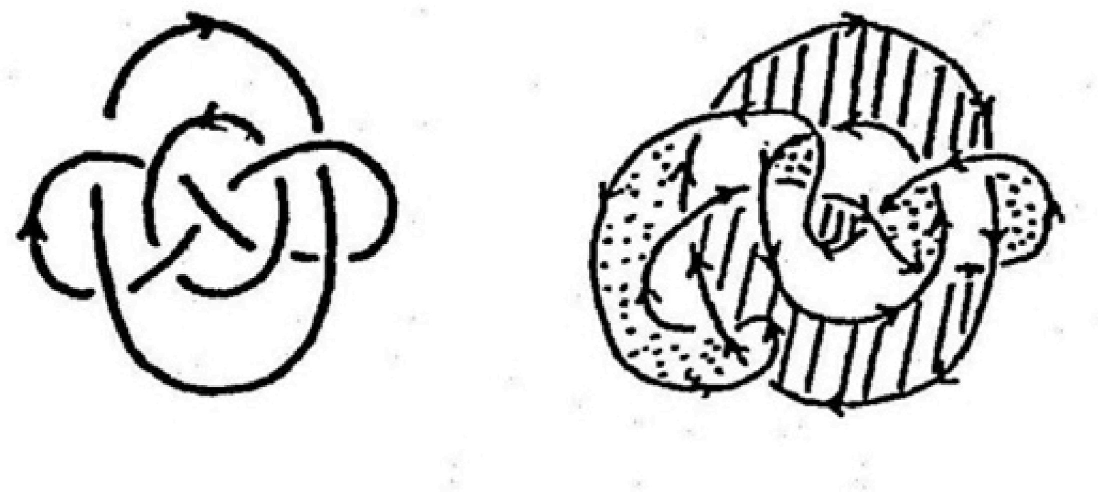
9^3_{19}③



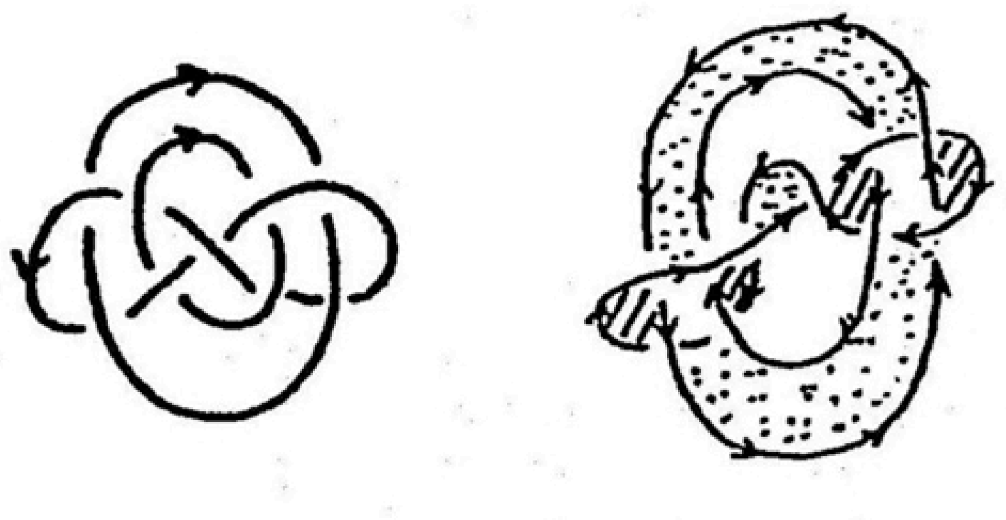
9^3_{19}④



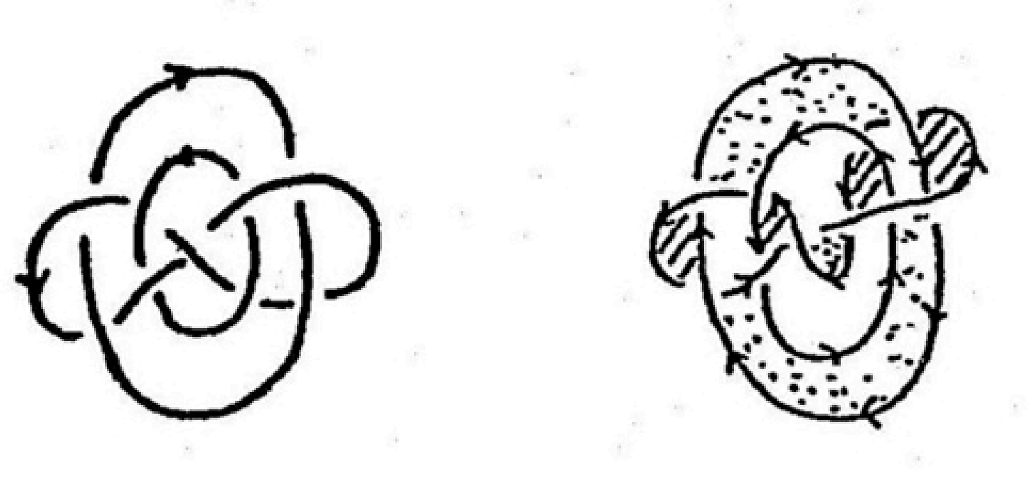
9^3_{20}①



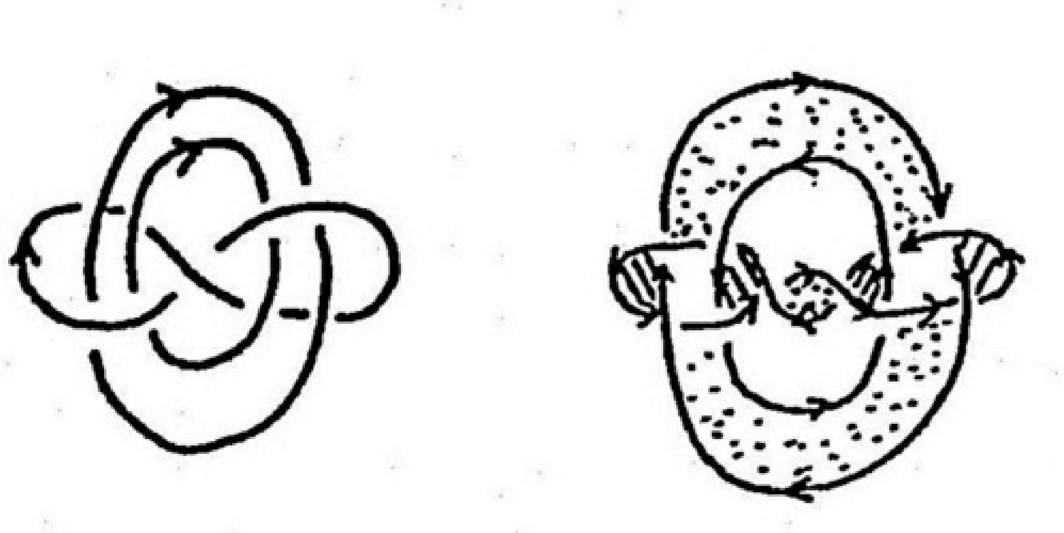
9^3_{20}②



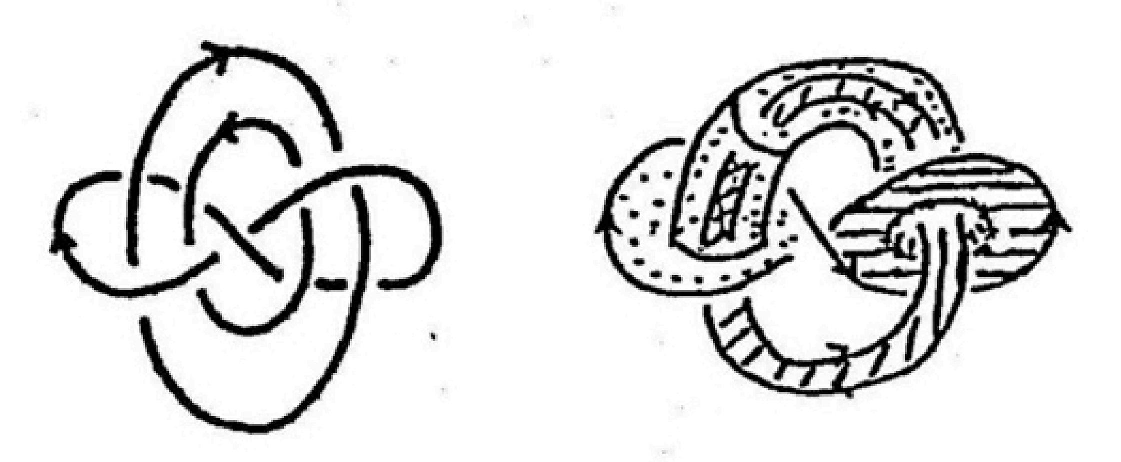
9^3_{20}③



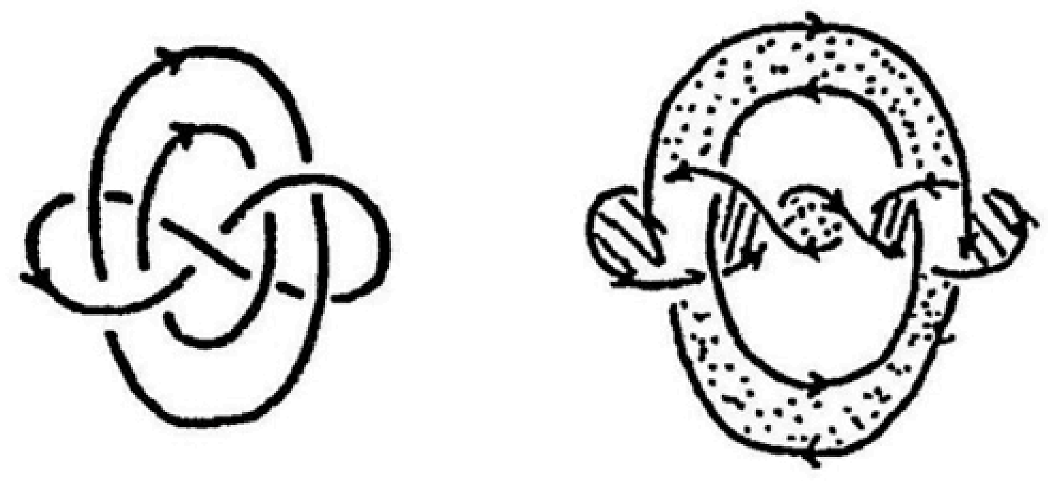
9^3_{20}④



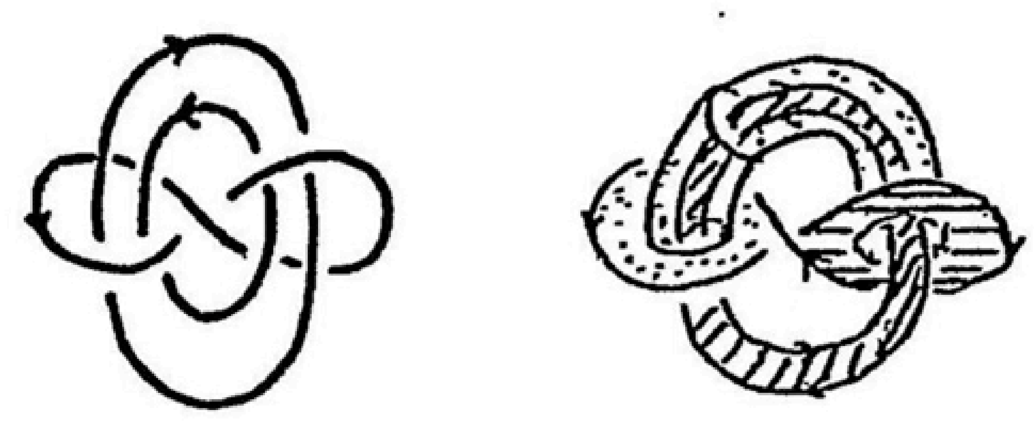
9^3_{21}①



9^3_{21}②

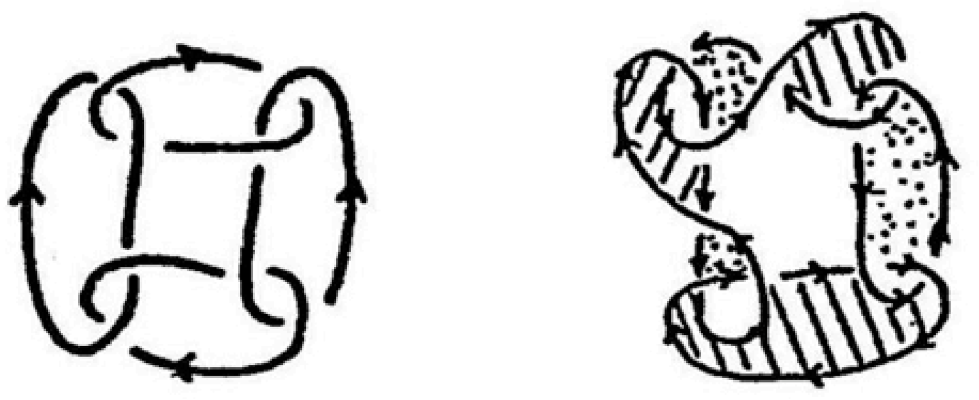


9^3_{21}③

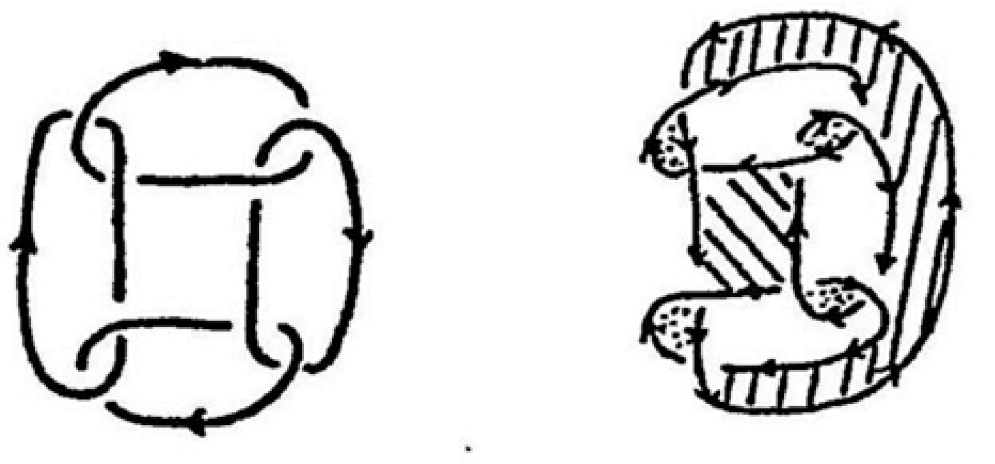


9^3_{21}④

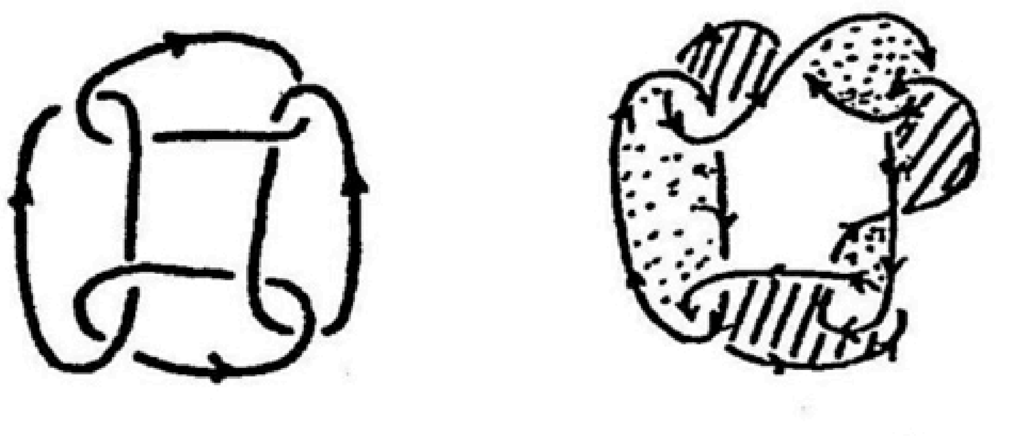
A.3 4-component prime links



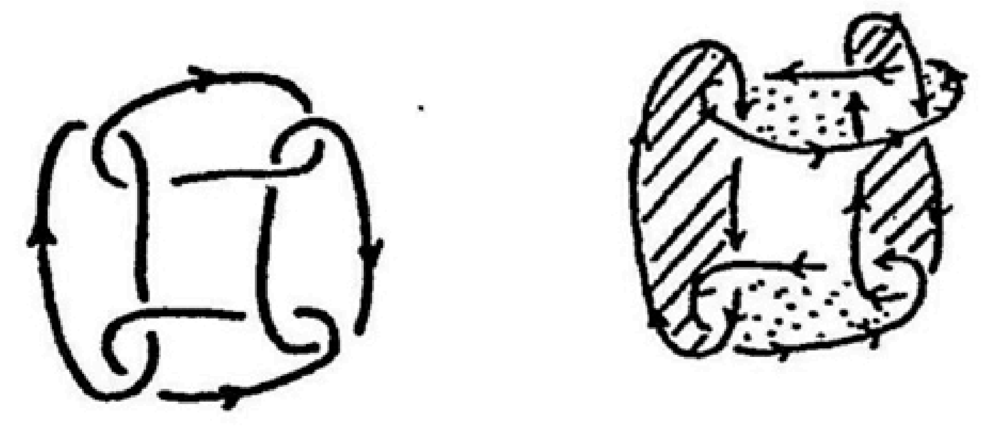
$8^4_1$①



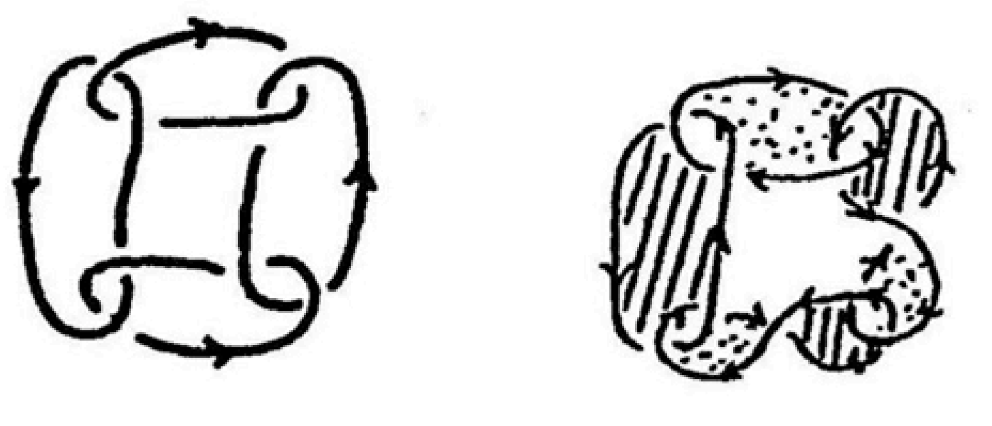
$8^4_1$②



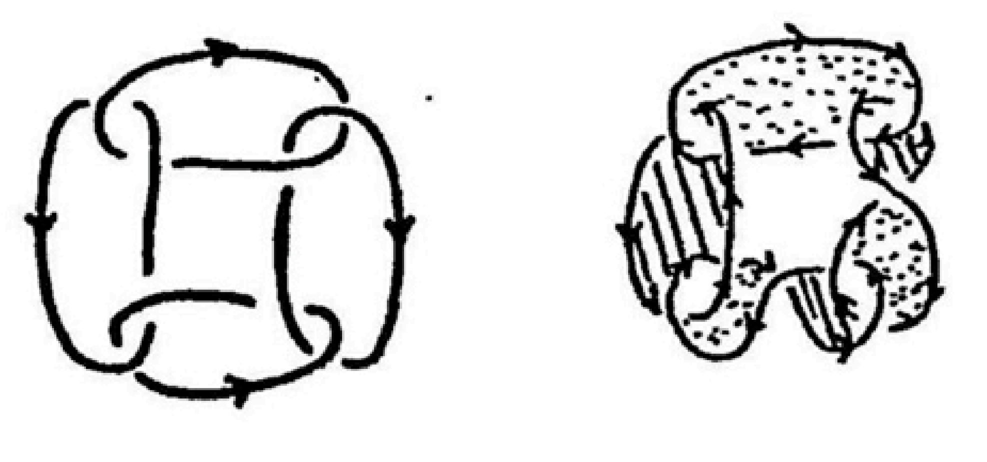
$8^4_1$③



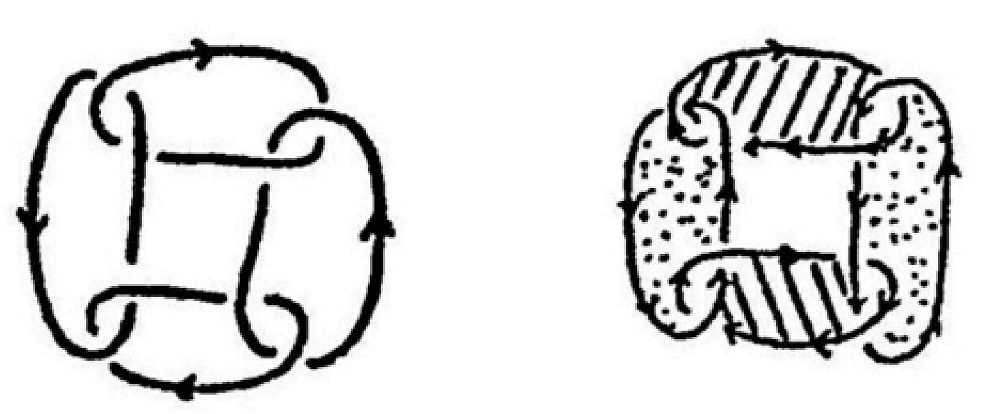
$8^4_1$④



$8^4_1$⑤

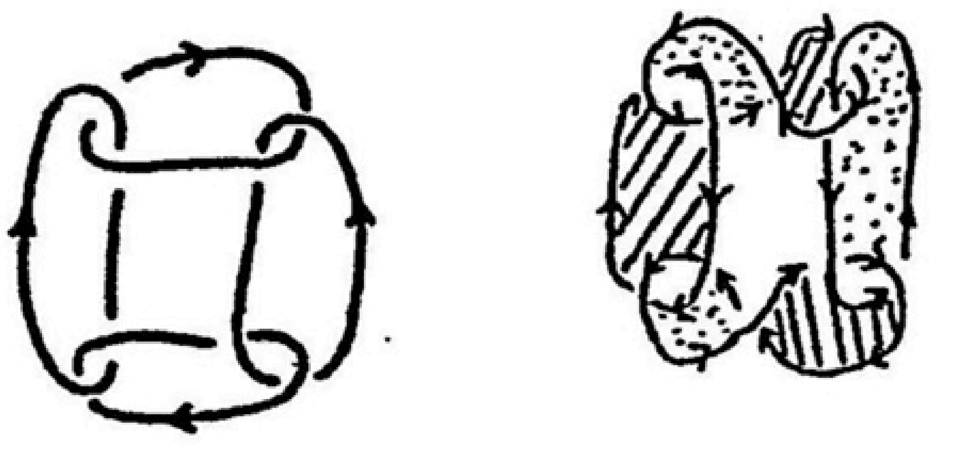


$8^4_1$⑥

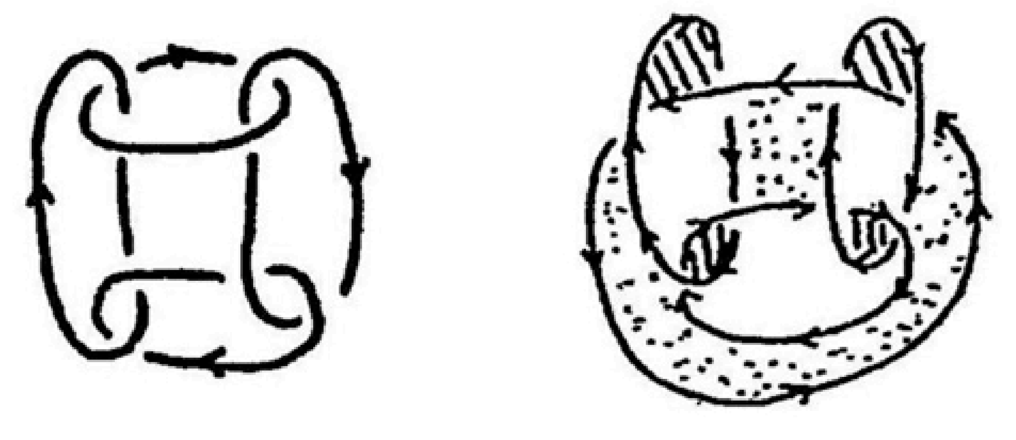


$8^4_1$⑦

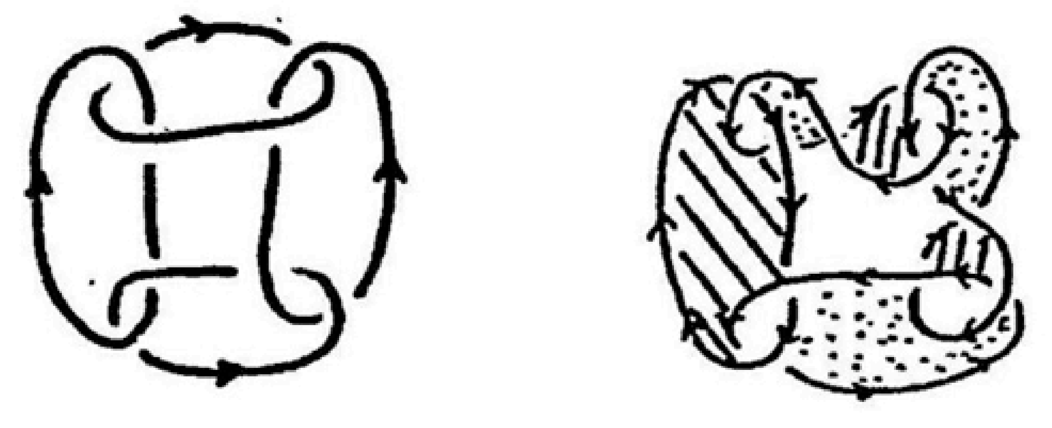
$8^4_1$⑧



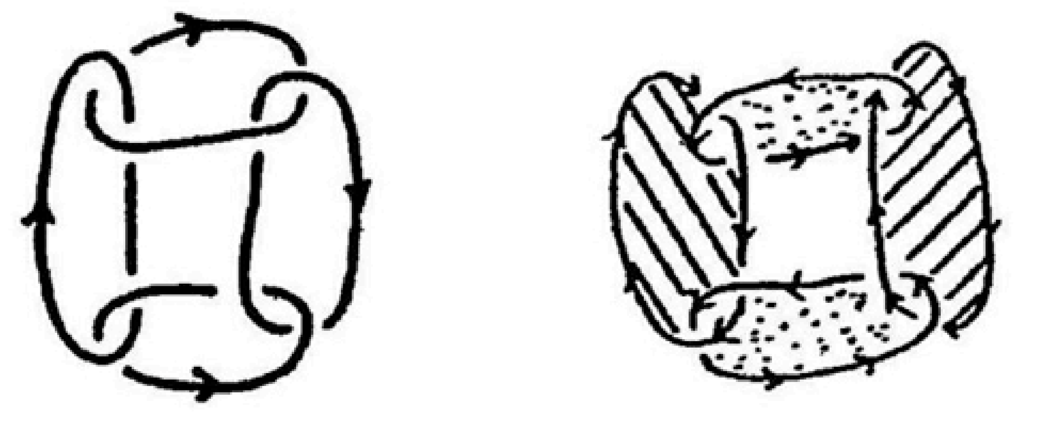
$8^4_2$①



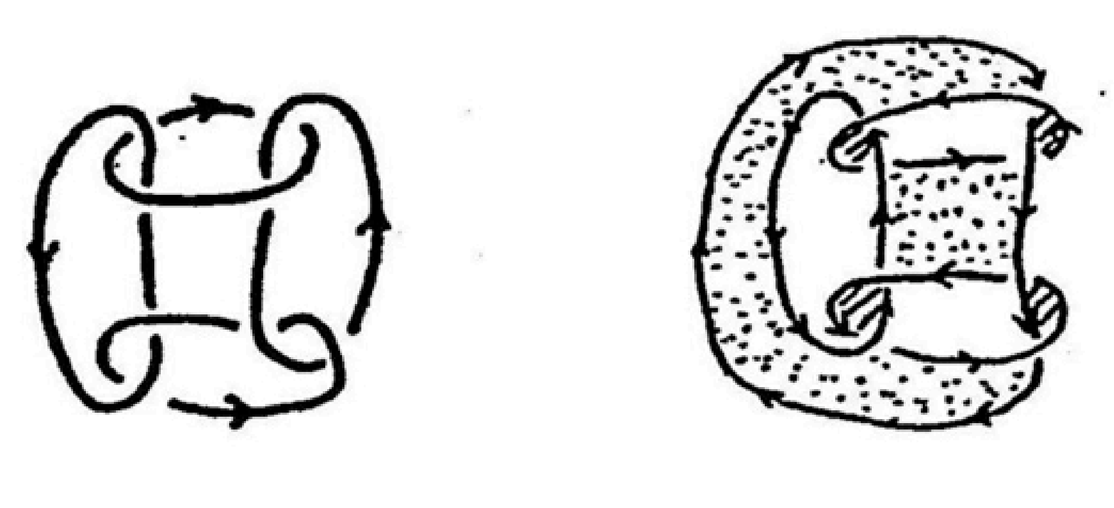
$8^4_2$②



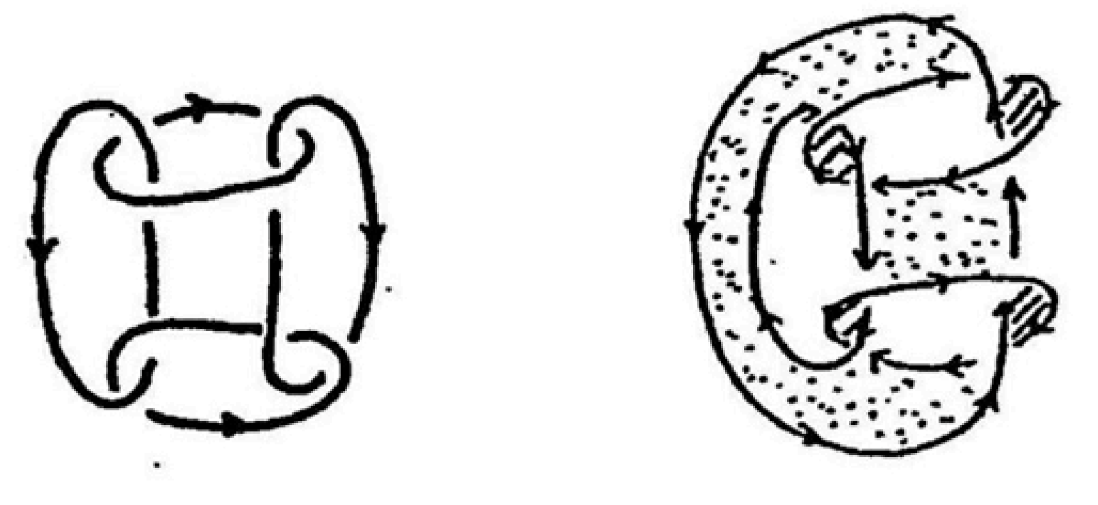
$8^4_2$③



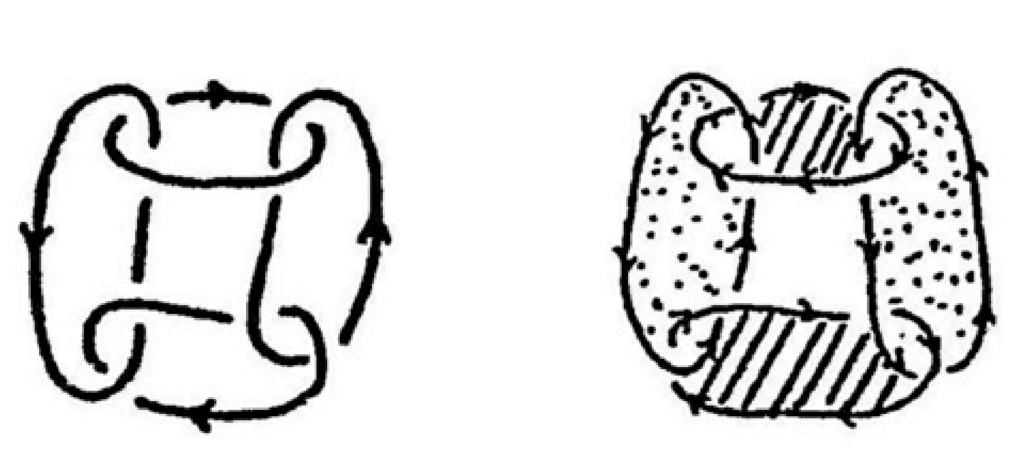
$8^4_2$④



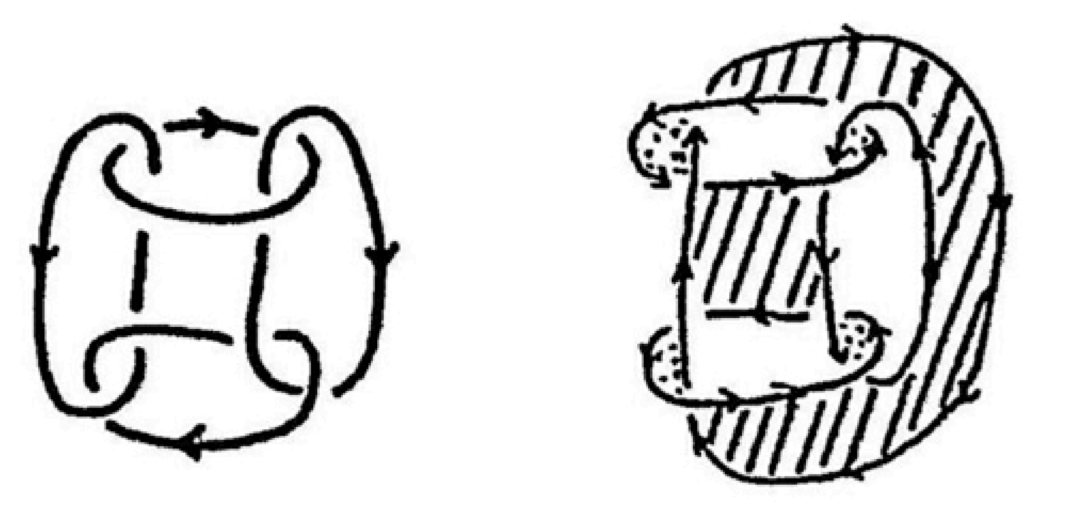
$8^4_2$⑤



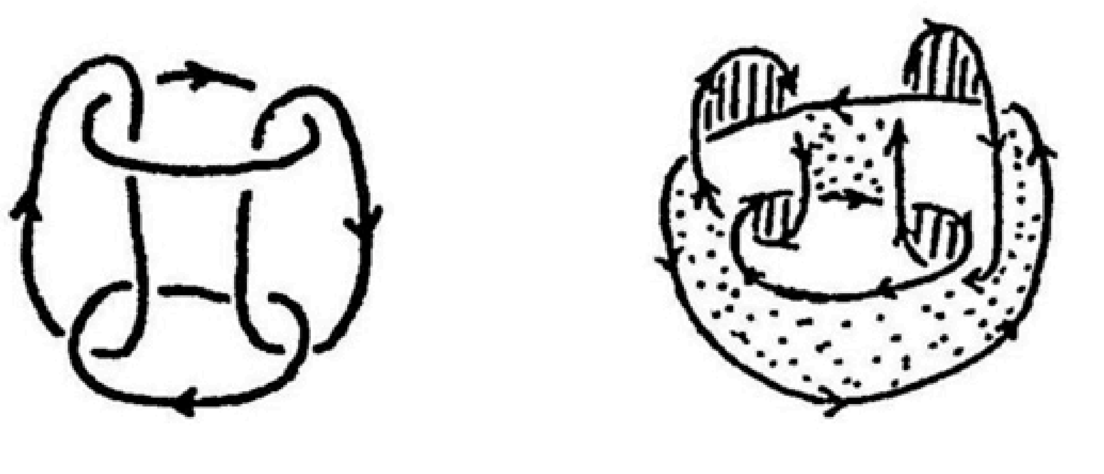
$8^4_2$⑥



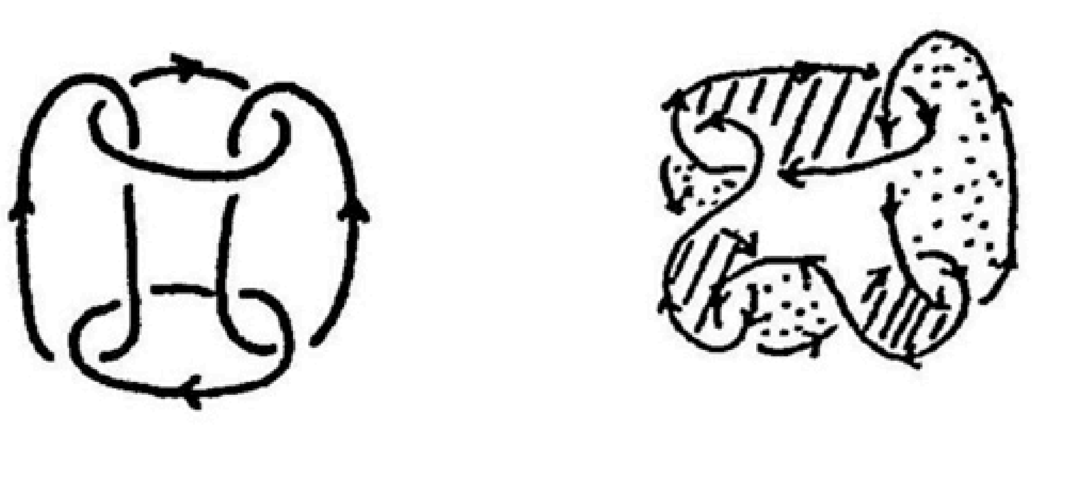
$8^4_2$⑦



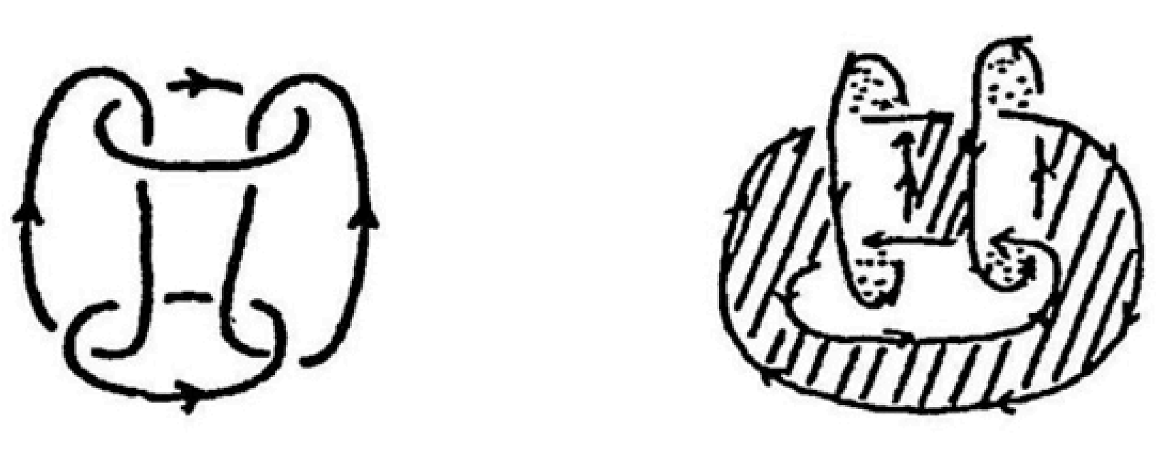
$8^4_2$⑧



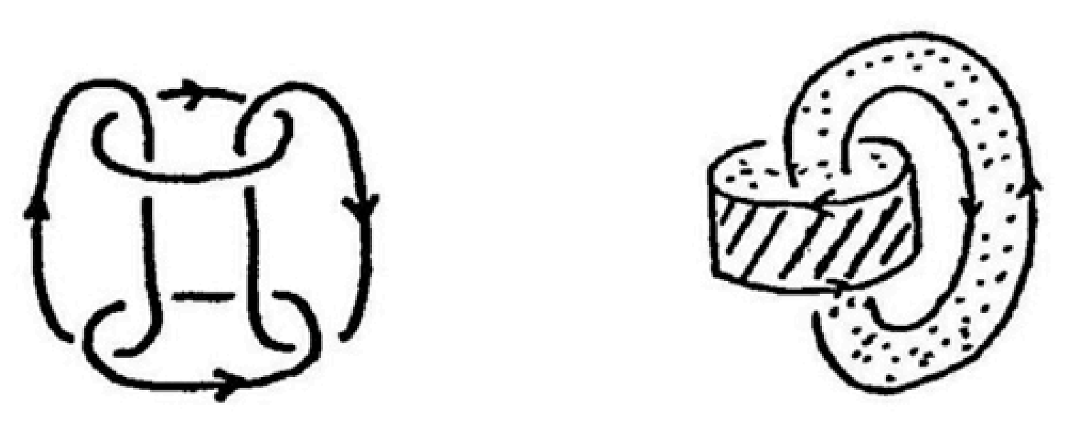
$8^4_3$①



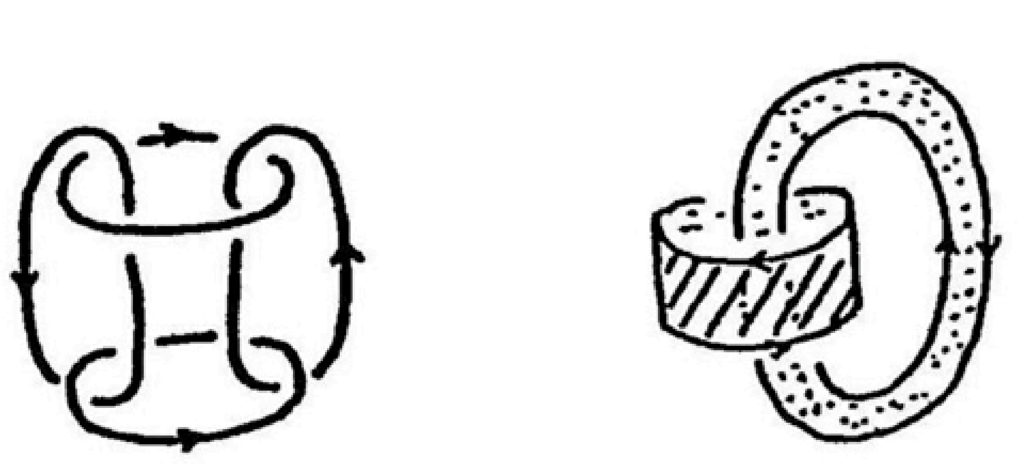
$8^4_3$②



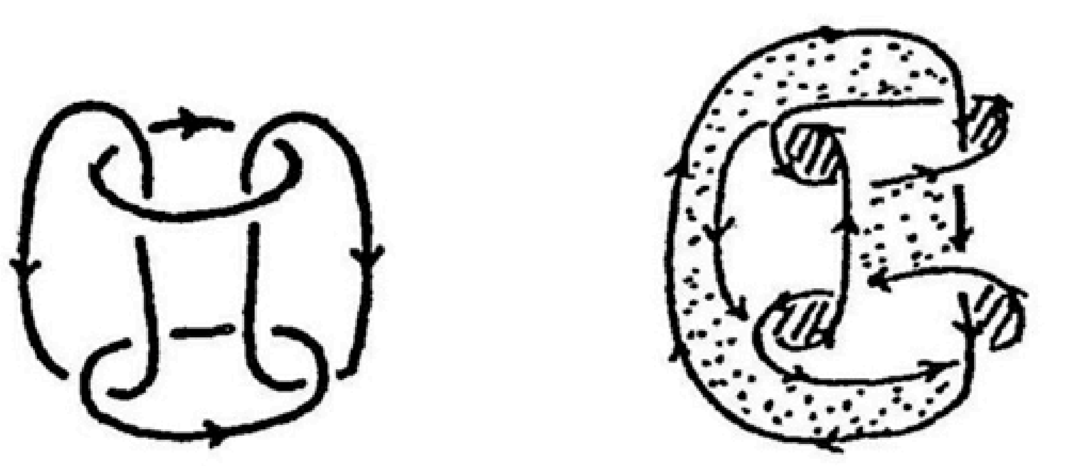
$8^4_3$③



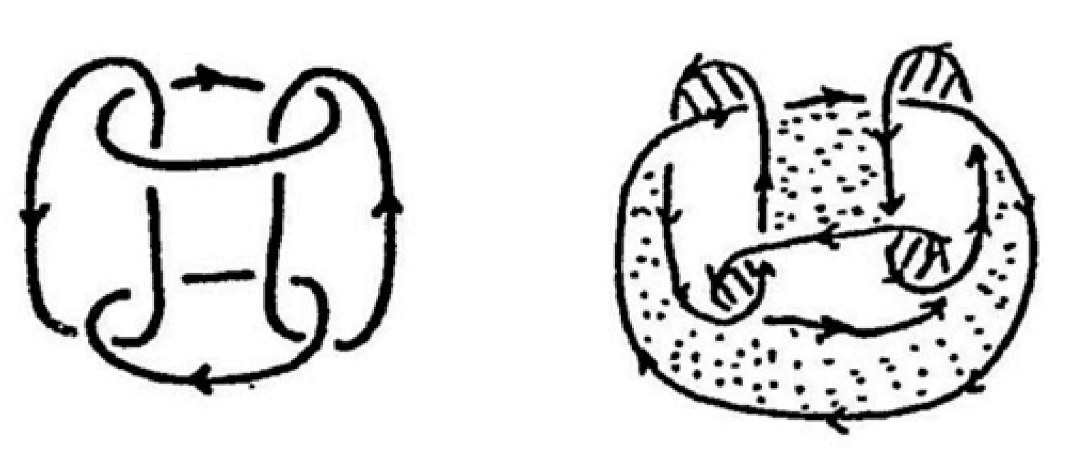
$8^4_3$④



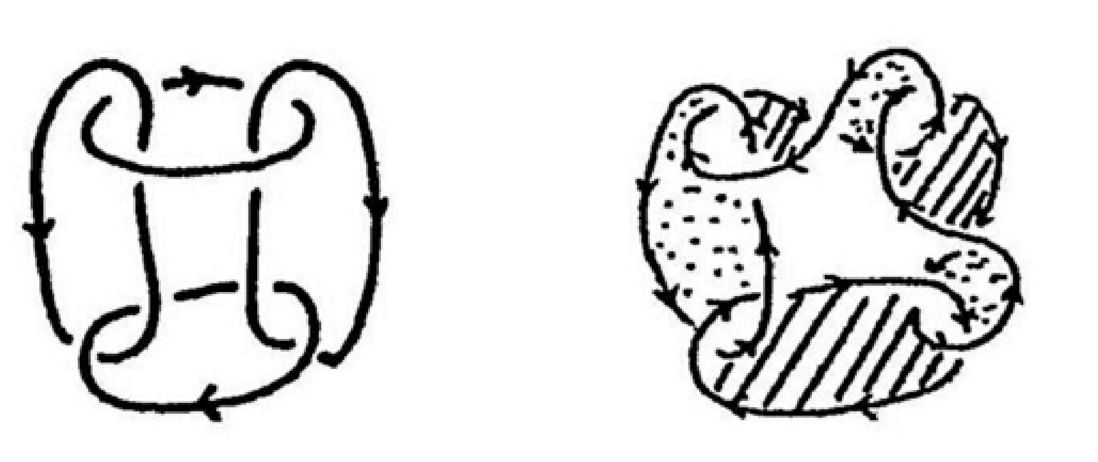
$8^4_3$⑤



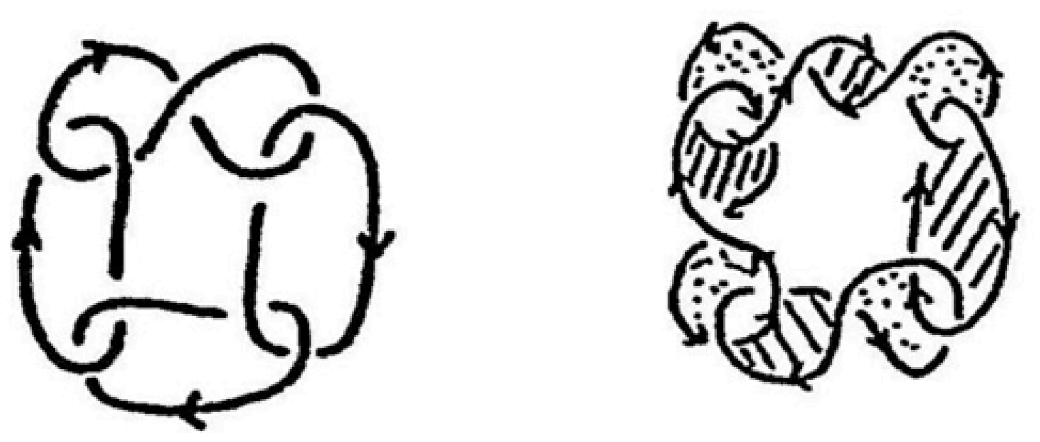
$8^4_3$⑥



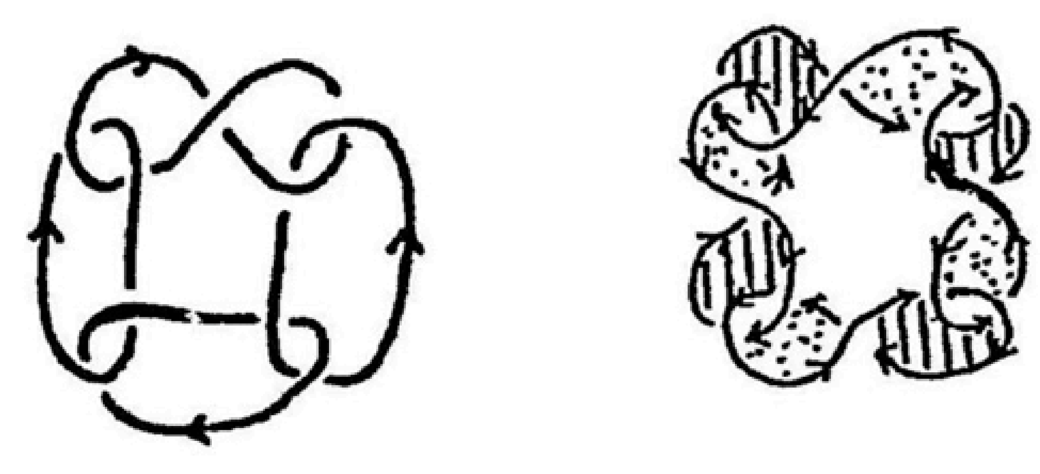
$8^4_3$⑦



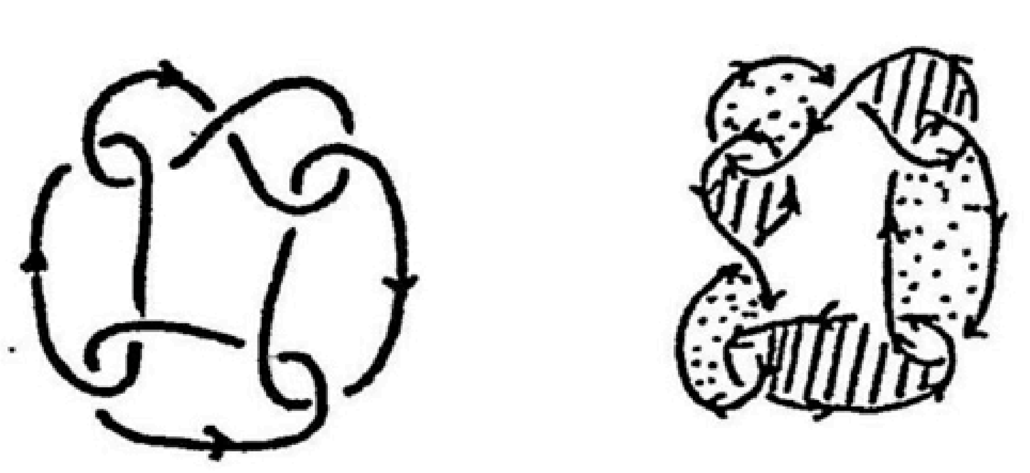
$8^4_3$⑧



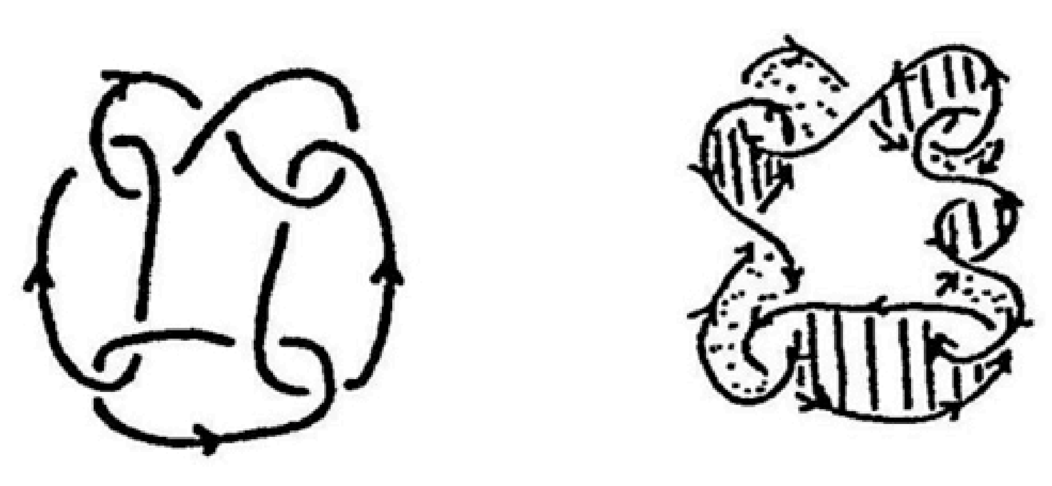
$9^4_1$①



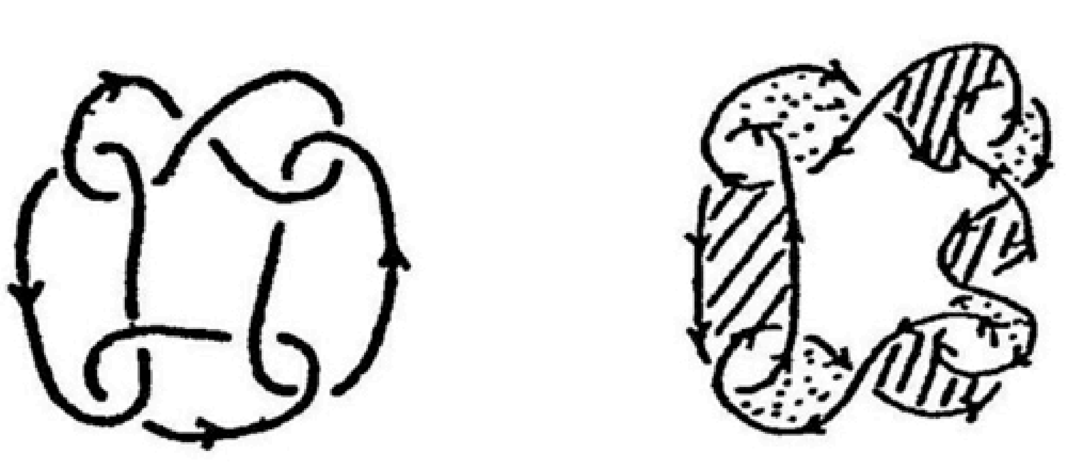
$9^4_1$②



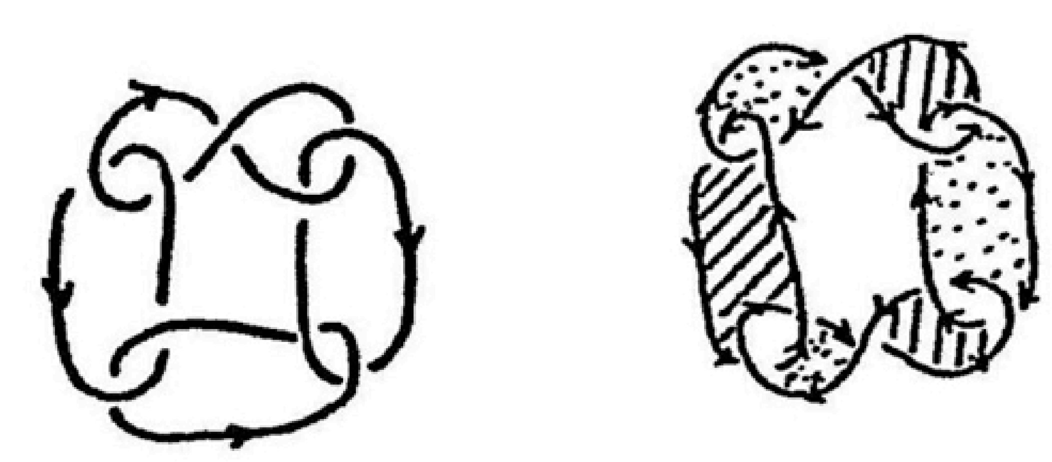
$9^4_1$③



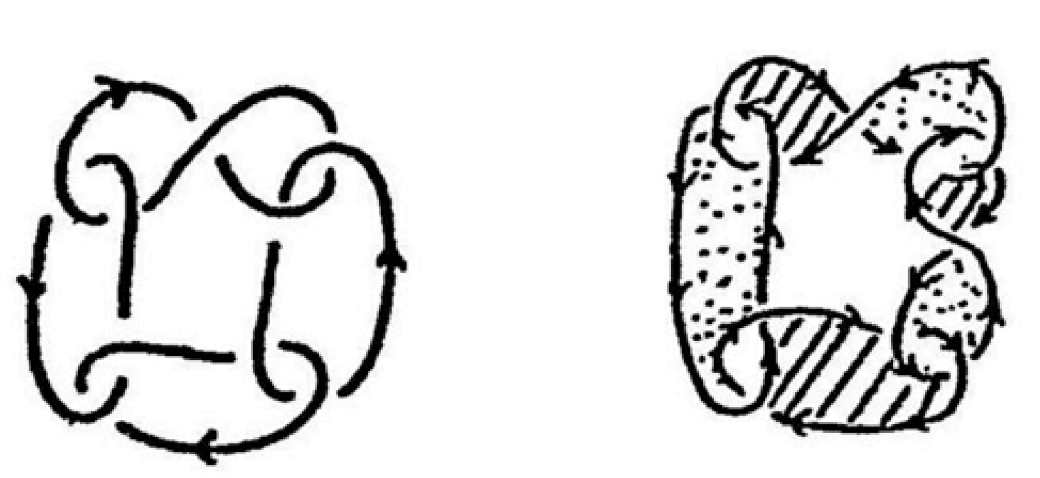
$9^4_1$④



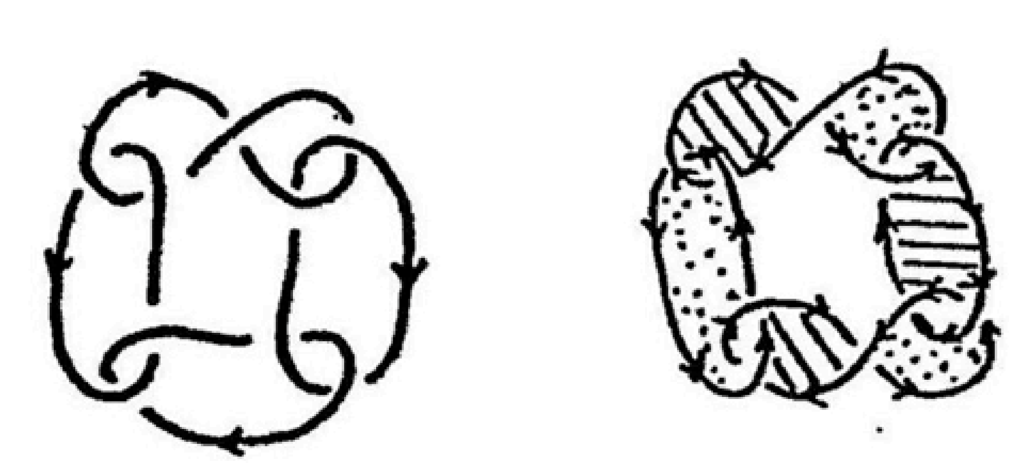
$9^4_1$⑤



$9^4_1$⑥



$9^4_1$⑦



$9^4_1$⑧

References

[1] C.C. Adams: *The knot book*, W.H.Freeman and Company (1994).

[2] S. Akbulut and R. Kirby: *Branched covers of surfaces in 4-manifolds*, Math. Ann. 252 (1980), 111-131.

[3] R.P.Anstee, J.H.Przytycki and D.Rolfsen : *Knot polynomials and generalized mutation*, Top. Appl. 32 (1989), 237-249.

[4] F. Bonahon and J. P. Otal: *Scindements de Heegaard des espaces lenticulaires*, Ann. Sci. Ec. Norm. Sup. 16 (1983), 451-466.

[5] R.I.Brooks, C.A.B.Smith, A.H.Stone and W.Tutte : *The dissection of rectangles into squares*, Duke Math.J. 7 (1940), 312-340.

[6] A. J. Casson and C. M. Gordon: *Reducing Heegaard splittings*, Topology. Appl. 27 (1987), 275-283.

[7] J. H. Conway: *An enumeration of knots and links and some of their related properties*, Computational problems in Abstract Algebra, Proc. Conf. Oxford 1967, Pergamon Press (1970), 329-358.

[8] M. Dąbkowski, M. Ishiwata, J. Przytycki and A. Yasuhara, *Rotation and signature invariant*, submitted.

[9] J. Dymara, T. Januszkiewicz, J. H. Przytycki: *Symplectic structure on Colorings, Lagrangian tangles and Tits buildings*, preprint (May 2001).

[10] P.Freyed, D. Yetter, J. Hoste, W.B.R.Lockorish, K.Millet and A.Ocneanu: *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) 12 (1985), 239-246.

[11] D.Gabai: *Foliations and genera of links*, Topology 23 (1984), 381-394.

[12] D.Gabai: *Detecting fibred links in S^3*, Com. Math. Helv. 61 (1986), 519-555.

[13] C.M. Gordon and R.A.Litherland : *On the signature of a link*, Invent. Math. 47 (1978), 53-69.

[14] H. Hilden, M. T. Lozano and J. M. Montesinos-Amilibia: *The arithmeticity of the figure eight knot orbifolds*, Topology '90 (Columbus, OH, 1990), 169–183, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.

[15] M. Ishiwata, J.H.Przytycki and A.Yasuhara : *Branched covers of tangles in three-balls*, Can. Math. Bull. 46 (2003), 356-364.

[16] W. Jaco: *Lectures on three manifold topology*, Conference board of Math. 43, A.M.S. (1980).

[17] G.T.Jin and D.Rolfsen : *Some remarks on rotors in link theory*, Can. Math.Bull. 34 (1991), 480-484.

[18] V.F.R.Jones : *Commuting transfer matrices and link polynomials*, Internat. J. Math. 3 (1992), 205-212.

[19] T.Kanenobu : *Infinitely many knots with the same polynomial invariant*, Proc. Amer. Math. Soc. 97 (1986), 158-161.

[20] T.Kanenobu : *The homfly and Kauffman bracket polynomials for the generalized mutant of a link*, Topology Appl. 61 (1995).

[21] L.H.Kauffman : *State models and the Jones polynomial*, Topology 26 (1987), 395-407.

[22] L.H.Kauffman : *On Knots*, Ann. of Math. Studies 115, Princeton Univ. Press (1987).

[23] A. Kawauchi: *A survey of knot theory*, Birkhauser Verlag, Basel (1996).

[24] K. Kodama : *KNOT (Software)*, http://www.math.kobe-u.ac.jp/~kodama/e-index.html.

[25] J. Levine : *Polynomial invariants of knots of codimension 2*, Ann. of Math. 84 (1966), 537-544.

[26] W.B.R. Lickorish : *An Introduction to Knot Theory*, Graduate Texts in Mathematics 175, Springer (1997).

[27] J. M. Montesinos: *Variedades de Seifert que son recubridadores ciclicos ramificados de dos hojas*, Bol. Soc. Mat. Mexicana 18 (1973), 1-32.

[28] J. M. Montesinos: *Surgery on links and double branched covers of S^3, Knots, groups and 3-manifolds*, Ann. Math. Studies, 84, Princeton Univ. Press. (1975), 227-259.

[29] K. Murasugi: *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965), 387-422.

[30] K. Murasugi: *On the signature of links*, Topology 9 (1970), 283-298.

[31] V. V. Prasolov and A. B. Sossinsky: *Knots, links, braids and 3-manifolds*, A.M.S. (1997).

[32] J.H.Przytycki : *Search for different links with the same Jones' type polynomials: Ideas from graph theory and statistical mechanics*, Panoramas of Mathematics, Banach Center Publications 34 (1995), 387-422.

[33] J.H.Przytycki and P.Traczyk: *Invariants of links of Conway type*, Kobe J. Math. 4 (1988), 115-139.

[34] D.Rolfsen: *Knots and links*, Mathematics Lecture Series, No.7, Publish or Perish Inc., Berkeley, Calif. (1976).

[35] D. Rolfsen : *Global mutation of knots*, J. Knot Theory Ramifications, 3 (1994), 407-417.

[36] D. Rolfsen: *Maps between 3-manifolds with nonzero degree: a new obstruction*, in preparation for Proceedings of New Techniques in Topological Quantum Field Theory, NATO Advanced Research Workshop, August 2001, Canada.

[37] P.Traczyk: *A note on rotant links*, J.Knot Theory Ramifications 8 (1999), 397-403.

[38] P. Traczyk: *Conway polynomial and oriented rotant links*, preprint (2001).

[39] A.G.Tristram: *Some cobordism invariants for links*, Proc. Camb. Phil. Soc. 66 (1969), 251-264.

[40] H.F.Trotter : *Homology of group system with applications to knot theory*, Ann. of Math. 76(2) (1962), 464-498.

Acknowledgements.

The author is deeply grateful to Professor Akio Kawauchi for giving me an opportunity to write this thesis and giving me much valuable suggestions. The author is also much thankful to Professor Taizo Kanenobu for valuable comments at Meetings of Knot Theory.

The author wishes to express her gratitude to Professor Kazuaki Kobayashi for his constant encouragement for these 12 years since she began to study Knot Theory, and also wishes to thank Professor Yoshiyuki Ohyama for his encouragement at Tokyo Woman's Christian University.

The work of this thesis was partially done during the stay of the author at the George Washington University. The author especially wishes to thank Professors Józef H.Przytycki, Akira Yasuhara and Mieczysław Dąbkowski for their valuable disscusions, much suggestions and their hospitality during her visit and until now.

The author is thankful to Professor Hiroshi Goda who gave us an open problem at TWCU Topology Monthly Seminar, which is related to Appendix of this thesis and also gave her valuable suggestions for her research.

The author is much thankful to the faculty of Department of Mathematics of Tokyo Woman's Christian University and the George Washington University for their hearty encouragements and supports for her research activities on Knot Theory.

List of papers by Makiko Ishiwata

1. Book presentation of the complete n-partite graphs, Kobe Journal of Mathematics, 13 (1996) 27-48.

2. H-path extendability of 3-regular graphs, (with Eri Takahashi), Science reports of Tokyo Woman's Christian University, (1999) 1523-1531.

3. A surgery description of an integral homology three-sphere respecting the Casson invariant, Proceeding of Topology Meeting 'Mathematics of Knots', (1999) 149-156, in Japanese.

4. A surgery description of integral homology three-sphere respecting the Casson invariant, Journal of Knot Theory and Its Ramifications, 10(6) (2001) 907-911.

5. Branched covers of tangles in three-balls,(with Józef H. Przytycki and Akira Yasuhara), Proceedings of Topology meeting 'The Present, the Past and the Future of Knots', (2002) 157-164, in Japanese.

6. Branched covers of tangles in three-balls, (with Józef H.Przytycki and Akira Yasuhara), Can. Math. Bull. 46 (2003), 356-364.

7. The signatures of rotant links, (with Mieczysław Dąbkowski, Józef H.Przytycki and Akira Yasuhara), Proceedings of Topology meeting 'Topology of Knots V',(2003) 158 - 164, in Japanese.